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Ch. 7: Convergence of Infinite Sequences

S1 Convergence of infinite sequences of reals

Def. An infinite sequence is any function, f , with domain \mathbb{N} . We usually write the infinite sequence f as $\langle f(n) \rangle_{n \in \mathbb{N}}$.

Def. An infinite sequence of reals is any function, f , with domain \mathbb{N} and $\text{ran}(f) \subseteq \mathbb{R}$.

Ex. 1 Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be defined by $f(n) = n/(n+2)$. Then f is an infinite sequence of reals. We can write this sequence as $\langle \frac{n}{n+2} \rangle_{n \in \mathbb{N}}$

$$= \left\langle \frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \dots \right\rangle$$

Now if we look at the first few terms, $\frac{n}{n+2}$ might not appear to approach any number, but if we look at the term with $n=98$ we see that

$$\frac{n}{n+2} = \frac{98}{100} = 0.98 \quad \text{and if we look at the term with } n=998, \text{ we see that } \frac{n}{n+2} = \frac{998}{1000} = 0.998.$$

So it now appears that $\frac{n}{n+2}$ approaches 1 as n gets larger & larger. We will say that $\langle \frac{n}{n+2} \rangle_{n \in \mathbb{N}}$ "converges" to 1 and also will write

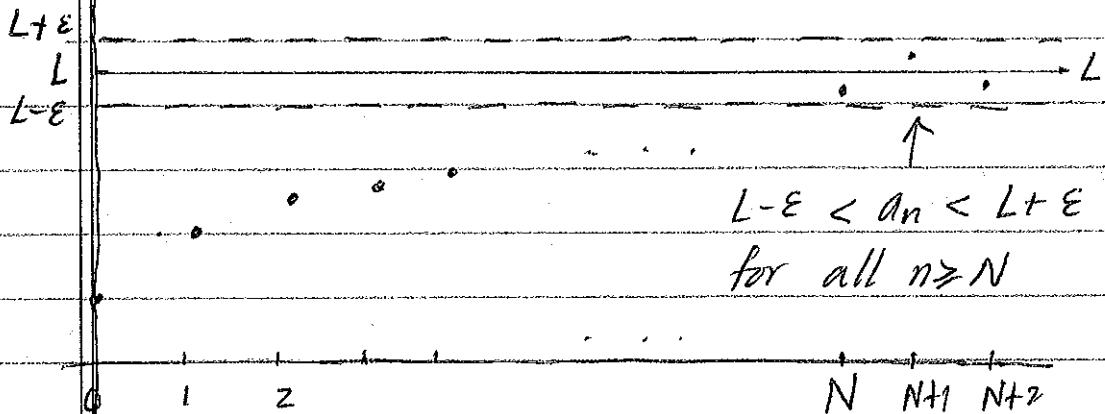
$$\lim_{n \rightarrow \infty} \frac{n}{n+2} = 1, \text{ but first we need to make the notion of convergence precise.}$$

Def. The sequence of reals $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to the real number L if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N} \text{ with } n \geq N) (|a_n - L| < \varepsilon).$$

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Def The sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent if we can find a real number L such that $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to L . The sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is divergent if it is not convergent.



Ex. 2 Let $a_n = \frac{n}{n+2}$. Prove that $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to $L=1$.

Sol. Let $\varepsilon > 0$ be given. Choose $N = \lfloor 2/\varepsilon \rfloor + 1$. Then $N > 2/\varepsilon$. So $N/2 > 1/\varepsilon$. $\therefore 2/N < \varepsilon$.

(Here $\lfloor 2/\varepsilon \rfloor = \text{smallest integer } \geq 2/\varepsilon$). So for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$\begin{aligned}|a_n - L| &= \left| \frac{n}{n+2} - 1 \right| = \left| \frac{n - (n+2)}{n+2} \right| \\&= \left| \frac{-2}{n+2} \right| = \frac{2}{n+2} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.\end{aligned}$$

$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N} \text{ with } n \geq N)(|a_n - L| < \varepsilon)$. Hence $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to L .

Ex. 3 Let $a_n = \frac{3[n+(-1)^n]}{2n+1}$. Prove that $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to $L = \frac{3}{2}$.

Sol. Let $\varepsilon > 0$ be given. Choose $N = \lfloor 9/4\varepsilon \rfloor + 1$. Then $N > 9/4\varepsilon$. So $4N/9 > 1/\varepsilon$. $\therefore 9/4N < \varepsilon$.

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Ex.3 So for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$\begin{aligned} |a_n - L| &= \left| \frac{3 \cdot [3n + (-1)^n]}{2n+1} - \frac{3}{2} \right| \\ &= \left| \frac{2[3n + 3(-1)^n] - 3(2n+1)}{2(2n+1)} \right| \\ &= \left| \frac{6 \cdot (-1)^n - 3}{2(2n+1)} \right| \leq \frac{9}{4n+2} \\ &< \frac{9}{4n} < \frac{9}{4N} < \varepsilon \end{aligned}$$

$\therefore (\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall n \in \mathbb{N} \text{ with } n \geq N) (|a_n - L| < \varepsilon)$

Hence $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to L .

Ex.4 Let $a_n = \langle 3 \rangle_{n \in \mathbb{N}}$. Prove that $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to $L = 3$.

Sol. Let $\varepsilon > 0$ be given. Choose $N = 0$. Then for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$\begin{aligned} |a_n - L| &= |3 - 3| \\ &= 0 < \varepsilon \end{aligned}$$

$\therefore (\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall n \in \mathbb{N} \text{ with } n \geq N) (|a_n - L| < \varepsilon)$

$\therefore \langle 3 \rangle_{n \in \mathbb{N}}$ converges to 3.

Ex.5 Let $a_n = \langle \frac{1}{2n+3} \rangle_{n \in \mathbb{N}}$. Prove that $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent.

Sol. Let $L = 0$. Now let $\varepsilon > 0$ be given. Choose $N = \lfloor \frac{1}{2\varepsilon} \rfloor + 1$.

Then $N > \frac{1}{2\varepsilon}$. So $2N > \frac{1}{\varepsilon}$. $\therefore \frac{1}{2N} < \varepsilon$.

Now for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$|a_n - L| = \left| \frac{1}{2n+3} - 0 \right| = \frac{1}{2n+3} < \frac{1}{2N} \leq \frac{1}{2N} < \varepsilon.$$

$\therefore (\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall n \in \mathbb{N} \text{ with } n \geq N) (|a_n - L| < \varepsilon)$

$\therefore \langle \frac{1}{2n+3} \rangle_{n \in \mathbb{N}}$ converges to 0. $\therefore \langle \frac{1}{2n+3} \rangle_{n \in \mathbb{N}}$ is convergent.

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Prop.1 If $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to L_1 , and $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to L_2 , then $L_1 = L_2$. (In other words, an infinite sequence can converge only to a single number.)

Proof: Suppose $L_1 \neq L_2$. Then $|L_1 - L_2| > 0$. Put $\varepsilon = |L_1 - L_2|/2$. Since $\langle a_n \rangle$ converges to L_1 , and $\varepsilon > 0$, we can find an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n - L_1| < \varepsilon$.

Also since $\langle a_n \rangle$ converges to L_2 and $\varepsilon > 0$, we can find an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have

$$|a_n - L_2| < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_N) + (a_N - L_2)| \\ &\leq |L_1 - a_N| + |a_N - L_2| \\ &= |a_N - L_1| + |a_N - L_2| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon = |L_1 - L_2|. \end{aligned}$$

$\therefore |L_1 - L_2| < |L_1 - L_2|$, a contradiction.

Hence we cannot have $L_1 \neq L_2$. $\therefore L_1 = L_2$.

Def.(a) An infinite sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is bounded above if we can find a real number U such that $(\forall n \in \mathbb{N}) (a_n \leq U)$.

(b) $\langle a_n \rangle_{n \in \mathbb{N}}$ is bounded below if we can find a real number L such that $(\forall n \in \mathbb{N}) (L \leq a_n)$.

(c) An infinite sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is bounded if it is bounded both above and below. In other words, if we can find $L, U \in \mathbb{R}$ such that $(\forall n \in \mathbb{N}) (L \leq a_n \leq U)$.

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- Ex. 6 (a) Show that $\langle (5 - n/2) \rangle_{n \in \mathbb{N}}$ is bounded above
 (b) Show that $\langle (3n - 2) \rangle_{n \in \mathbb{N}}$ is bounded below.
 (c) Show that $\langle 4 \cdot (-1)^n + \frac{1}{n+1} \rangle_{n \in \mathbb{N}}$ is bounded.

Sol. (a) Let $U = 5$. Then for all $n \in \mathbb{N}$, we have

$$5 - n/2 \leq 5 - 0 = U.$$

So $(\forall n \in \mathbb{N}) (5 - n/2 \leq U)$. $\therefore \langle 5 - \frac{n}{2} \rangle_{n \in \mathbb{N}}$ is bounded above.

(b) Let $L = -2$. Then for all $n \in \mathbb{N}$, we have

$$3n - 2 \geq 0 - 2 = L.$$

$\therefore (\forall n \in \mathbb{N}) (L \leq 3n - 2)$. So $\langle 3n - 2 \rangle_{n \in \mathbb{N}}$ is bounded below.

(c) Let $U = 5$ & $L = -4$. Then for all $n \in \mathbb{N}$ we have

$$L = -4 + 0 < 4(-1)^n + \frac{1}{n+1} \leq 4 + \frac{1}{n+1} = 5 = U$$

So $\langle 4 \cdot (-1)^n + \frac{1}{n+1} \rangle_{n \in \mathbb{N}}$ is bounded.

Prop. 2 If $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent, then $\langle a_n \rangle_{n \in \mathbb{N}}$ is bounded.

Proof: Since $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent, we can find an $A \in \mathbb{R}$ such that $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A . Let $\varepsilon = 1$. Since $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A & $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that $(\forall n \geq N)$

$$|a_n - A| < \varepsilon, \text{ i.e., } |a_n - A| < 1$$

So $(\forall n \geq N) (-1 < a_n - A < 1)$, i.e.

$$(\forall n \geq N) (A - 1 < a_n < A + 1)$$

Now let $U = \max \{a_0, a_1, \dots, a_{N-1}, A+1\}$

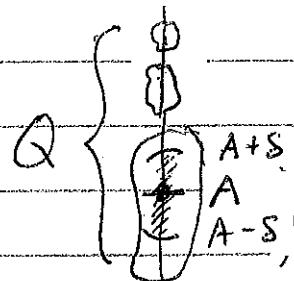
and $L = \min \{a_0, a_1, \dots, a_{N-1}, A-1\}$.

Then $(\forall n \in \mathbb{N}) (L \leq a_n \leq U)$. So $\langle a_n \rangle_{n \in \mathbb{N}}$ is bounded.

Note: $\langle (-1)^n \rangle_{n \in \mathbb{N}}$ is bounded, but not convergent.

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Def. A set of real numbers Q is said to be a neighbourhood of the real number A , if $(\exists \delta > 0) [(A-\delta, A+\delta) \subseteq Q]$.



Prop. 3 The infinite sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A \Leftrightarrow each nbhd Q of A contains all but a finite number of the terms of $\langle a_n \rangle_{n \in \mathbb{N}}$.

Proof. \Rightarrow) Suppose $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A . Let Q be any nbhd of A . Then we can find a $\delta > 0$, such that $(A-\delta, A+\delta) \subseteq Q$. Let $\varepsilon = \delta$. Since $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A and $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|a_n - A| < \varepsilon, \text{ i.e., } |a_n - A| < \delta.$$

So for all $n \geq N$, $A-\delta < a_n < A+\delta$. Hence $(\forall n \geq N) (a_n \in Q)$. So only a_0, \dots, a_{N-1} could possibly be outside of Q . Hence Q contains all but a finite no. of the terms of $\langle a_n \rangle_{n \in \mathbb{N}}$.

\Leftarrow) Suppose each nbhd of A contains all but a finite no. of the terms of $\langle a_n \rangle_{n \in \mathbb{N}}$. Let $\varepsilon > 0$. Since $Q = (A-\varepsilon, A+\varepsilon)$ is a nbhd of A , $(A-\varepsilon, A+\varepsilon)$ will contain all but a finite no. of the terms of $\langle a_n \rangle_{n \in \mathbb{N}}$. Let $N = \max \{n : a_n \notin (A-\varepsilon, A+\varepsilon)\} + 1$. Then for all $n \geq N$, $a_n \in (A-\varepsilon, A+\varepsilon)$. So $(\forall n \geq N) (|a_n - A| < \varepsilon)$. Hence $(\forall \varepsilon > 0) (\exists N \in \mathbb{N}) (\forall n \in \mathbb{N} \text{ with } n \geq N) (|a_n - A| < \varepsilon)$. Thus $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A .

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§2. Arithmetic operations on Convergent sequences.

Def.

Let $\langle a_n \rangle_{n \in \mathbb{N}}$ & $\langle b_n \rangle_{n \in \mathbb{N}}$ be infinite sequences.

We define (a) $c \cdot \langle a_n \rangle = \langle c \cdot a_n \rangle$ for each $c \in \mathbb{R}$

(b) $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$, (c) $\langle a_n \rangle - \langle b_n \rangle = \langle a_n - b_n \rangle$

(d) $\langle a_n \rangle \cdot \langle b_n \rangle = \langle a_n \cdot b_n \rangle$, & (e) $\langle a_n \rangle / \langle b_n \rangle = \langle a_n / b_n \rangle$
provided in (e) that b_n is never 0.

Theorem 4 Suppose $\langle a_n \rangle$ converges to A & $\langle b_n \rangle$ converges to B.

Then (a) $\langle a_n \rangle + \langle b_n \rangle$ converges to $A + B$

(b) $\langle a_n \rangle - \langle b_n \rangle$ converges to $A - B$

(c) $c \cdot \langle a_n \rangle$ converges to $c \cdot A$ for each $c \in \mathbb{R}$.

Proof. (a) Let $\epsilon > 0$ be given. Then $\epsilon/2 > 0$. Since

$\langle a_n \rangle$ converges to A and $\epsilon/2 > 0$, we can find an $N_1 \in \mathbb{N}$ such that ($\forall n \geq N_1$) $|a_n - A| < \epsilon/2$

Also since $\langle b_n \rangle$ converges to B and $\epsilon/2 > 0$,

we can find an $N_2 \in \mathbb{N}$ such that ($\forall n \geq N_2$) $|b_n - B| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then ($\forall n \geq N$)

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

So ($\forall \epsilon > 0$) ($\exists N \in \mathbb{N}$) ($\forall n \geq N$) ($|(a_n + b_n) - (A + B)| < \epsilon$).

Hence $\langle a_n + b_n \rangle$ converges to $A + B$.

(b) Do for H.W. (Hint: $|(a_n - b_n) - (A - B)| \leq |a_n - A| + |b_n - B|$.)

(c) There are two cases : (i) $c = 0$ & (ii) $c \neq 0$.

In case (i), $\langle c \cdot a_n \rangle = \langle 0 \rangle$ which clearly conv. to 0 = 0 · A.

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(c) In case (ii) $c \neq 0$, so $|c| \neq 0$. Let $\epsilon > 0$ be given. Then $\epsilon/|c| > 0$. Since $\langle a_n \rangle$ converges to A and $\epsilon/|c| > 0$, we can find an $N \in \mathbb{N}$ such that $(\forall n \geq N) |a_n - A| < \epsilon/|c|$. So $(\forall n \geq N)$

$$\begin{aligned} |c \cdot a_n - c \cdot A| &= |c(a_n - A)| \\ &= |c| \cdot |a_n - A| \\ &< |c| \cdot (\epsilon/|c|) = \epsilon. \end{aligned}$$

So $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N} \text{ with } n \geq N)(|c \cdot a_n - c \cdot A| < \epsilon)$. Hence $\langle c \cdot a_n \rangle$ converges to $c \cdot A$.

Theorem 5: Suppose $\langle a_n \rangle$ converges to A and $\langle b_n \rangle$ conv. to B . Then $\langle a_n \cdot b_n \rangle$ converges to $A \cdot B$.

Proof: Let $\epsilon > 0$ be given. Since $\langle a_n \rangle$ is convergent, it is bounded. So we can find an $M > 0$ such that $(\forall n \in \mathbb{N}) |a_n| \leq M$. Let $\epsilon' = \epsilon/(M+|B|)$.

Then $\epsilon' > 0$. Since $\langle a_n \rangle$ conv. to A & $\langle b_n \rangle$ conv. to B , we can find $N_1, N_2 \in \mathbb{N}$ such that

$$(\forall n \geq N_1) |a_n - A| < \epsilon'$$

$$(\forall n \geq N_2) |b_n - B| < \epsilon'$$

Let $N = \max \{N_1, N_2\}$. Then $(\forall n \geq N)$ we have

$$\begin{aligned} |(a_n \cdot b_n) - (A \cdot B)| &= |a_n \cdot b_n - a_n \cdot B + a_n \cdot B - A \cdot B| \\ &= |a_n(b_n - B) + (a_n - A) \cdot B| \\ &\leq |a_n| \cdot |b_n - B| + |(a_n - A) \cdot B| \\ &= |a_n| \cdot |b_n - B| + |B| \cdot |a_n - A| \\ &< M \cdot \epsilon' + |B| \cdot \epsilon' = (M+|B|)\epsilon' = \epsilon. \end{aligned}$$

So $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) |(a_n \cdot b_n) - (A \cdot B)| < \epsilon$

$\therefore \langle a_n \cdot b_n \rangle$ converges to $A \cdot B$.

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Lemma 6: Suppose $\langle b_n \rangle$ converges to B and $B \neq 0$. Then we can find an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $|b_n| \geq |B|/2$.

Proof: Let $\varepsilon = |B|/2$. Since $B \neq 0$, $\varepsilon > 0$. Also since $\langle b_n \rangle$ converges to B , we can find an $N \in \mathbb{N}$ such that for all $n \geq N_0$, we have

$$|b_n - B| < \varepsilon.$$

So for all $n \geq N_0$,

$$\begin{aligned} |b_n| &= |B - (B - b_n)| \\ &\geq |B| - |B - b_n| \\ &= |B| - |b_n - B| > |B| - |B|/2 = |B|/2. \end{aligned}$$

$$\therefore (\forall n \geq N_0) (|b_n| > |B|/2).$$

Hence if $\langle b_n \rangle$ conv to B , $(\exists N \in \mathbb{N}) (\forall n \geq N_0) (|b_n| > |B|/2)$.

Theorem 7: Suppose $\langle a_n \rangle$ & $\langle b_n \rangle$ converges to A & B , respectively and $\langle b_n \rangle$ is never 0 & $B \neq 0$. Then $\langle a_n \rangle / \langle b_n \rangle$ converges to A/B .

Proof: Let $\varepsilon > 0$ be given. Since $\langle b_n \rangle$ converges to B and $B \neq 0$, we can find an $N_0 \in \mathbb{N}$ such that $(\forall n \geq N_0) (|b_n| \geq |B|/2)$. let $\varepsilon' = \frac{\varepsilon \cdot |B|^2}{2(|A| + |B|)}$. Then $\varepsilon' > 0$.

Since $\langle a_n \rangle$ conv. to A & $\langle b_n \rangle$ conv. to B , we can find $N_1, N_2 \in \mathbb{N}$ such that

$$(\forall n \geq N_1) (|a_n - A| < \varepsilon')$$

$$\text{and } (\forall n \geq N_2) (|b_n - B| < \varepsilon').$$

Let $N = \max \{N_0, N_1, N_2\}$. Then for all $n \geq N$,

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$$\begin{aligned}
 \left| \frac{a_n}{b_n} - A \right| &= \left| \frac{a_n B - b_n A}{b_n B} \right| = \left| \frac{a_n B - AB + AB - b_n A}{b_n B} \right| \\
 &= \left| (a_n - A) \cdot B + A(B - b_n) \right| / |b_n \cdot B| \\
 &\leq \{ |(a_n - A) \cdot B| + |A \cdot (B - b_n)| \} / \{ |b_n| \cdot |B| \} \\
 &= \frac{|a_n - A| \cdot |B| + |b_n - B| \cdot |A|}{|b_n| \cdot |B|} \\
 &\leq \frac{\varepsilon' \cdot |B| + \varepsilon' \cdot |A|}{(|B|/2) \cdot |B|} \\
 &= \frac{2\varepsilon' \cdot (|A| + |B|)}{|B|^2} = \varepsilon
 \end{aligned}$$

So ($\forall \varepsilon > 0$) ($\exists N \in \mathbb{N}$) ($\forall n \geq N$) ($|(\frac{a_n}{b_n}) - (A/B)| < \varepsilon$)
 $\therefore \langle a_n/b_n \rangle$ converges to $A \cdot B$.

Theorem 9 (Squeezing Theorem). Suppose $\langle a_n \rangle$ conv. to 0 and $\langle b_n \rangle$ is a sequence such that ($\forall n \in \mathbb{N}$) ($b_n \in a_n$). Then $\langle b_n \rangle$ also converges to 0.

Proof: Let $\varepsilon > 0$ be given. Since $\langle a_n \rangle$ conv. to 0, we can find $n_0 \in \mathbb{N}$ such that ($\forall n \geq n_0$) ($|a_n| < \varepsilon$). So for all $n \geq n_0$,

$$\begin{aligned}
 |b_n - 0| = |b_n| &\leq |a_n| < \varepsilon. \\
 \therefore (\forall \varepsilon > 0) (\exists n_0 \in \mathbb{N} \text{ with } n \geq n_0) (|b_n - 0| < \varepsilon).
 \end{aligned}$$

Hence $\langle b_n \rangle$ converges to 0.

Ex. 1 Show that (a) $\sqrt{n+1} - \sqrt{n}$ converges to 0
(b) $\sqrt{n+\sqrt{n}} - \sqrt{n}$ converges to $1/2$.

Hints

- $\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) / (\sqrt{n+1} + \sqrt{n}) = 1 / (\sqrt{n+1} + \sqrt{n})$.
- $\sqrt{n+\sqrt{n}} - \sqrt{n} = (\sqrt{n+\sqrt{n}} - \sqrt{n})(\sqrt{n+\sqrt{n}} + \sqrt{n}) / (\sqrt{n+\sqrt{n}} + \sqrt{n}) = \sqrt{n} / (\sqrt{n+\sqrt{n}} + \sqrt{n})$.

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Cauchy sequences

Def. The infinite sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence if $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)$
 $|a_m - a_n| < \varepsilon$.

Prop. 10: If $\langle a_n \rangle$ is a convergent sequence, then $\langle a_n \rangle$ is a Cauchy sequence.

Proof: Suppose $\langle a_n \rangle$ is convergent. Then we can find an $A \in \mathbb{R}$ such that $\langle a_n \rangle$ converges to A . Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$. Since $\langle a_n \rangle$ converges to A and $\varepsilon/2 > 0$, we can find an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - A| < \varepsilon/2$$

So for all $m, n \geq N$, we have

$$\begin{aligned} |a_m - a_n| &= |a_m - A + A - a_n| \\ &\leq |a_m - A| + |A - a_n| \\ &= |a_m - A| + |a_n - A| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

So $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N) (|a_m - a_n| < \varepsilon)$.

Hence $\langle a_n \rangle$ is a Cauchy sequence.

Prop. 11: If $\langle a_n \rangle$ is a Cauchy sequence, then $\langle a_n \rangle$ is bounded.

Proof: Do for H.W. (Proof is very similar to that of Prop. 2)

Note: $\langle (-1)^n \rangle_{n \in \mathbb{N}}$ is bounded, but is not a Cauchy sequence.

Prop. 12 Suppose $\langle a_n \rangle$ & $\langle b_n \rangle$ are Cauchy sequences. Then

- (a) $\langle a_n \rangle + \langle b_n \rangle$ is also a Cauchy sequence.
- (b) $\langle a_n \rangle - \langle b_n \rangle$ is also a Cauchy sequence.
- (c) $c \cdot \langle a_n \rangle$ is also a Cauchy sequence.
- (d) $\langle a_n \rangle \cdot \langle b_n \rangle$ is also a Cauchy sequence.

Proof: Do for H.W. (Very similar to those of Theorems 4&5)

Def. Let S be a set of real numbers. We say that A is an accumulation point of S if every nbhd of A contains infinitely points of S .

Ex. 1 (a) Let $S_1 = \{(n-1)/n : n \in \mathbb{Z}^+\}$. Then 1 is the only accumulation point of S_1 . Notice that $1 \notin S_1$.

(b) Let $S_2 = \{(-1)^n/n : n \in \mathbb{Z}^+\}$. Then 0 is the only accumulation point of S_2 .

(c) Let $S_3 = \{2^{-n} + (-1)^n : n \in \mathbb{Z}^+\}$. Then 1 & -1 are the two accumulation points of S_3 .

Prop. 13 A is an accumulation point of $S \Leftrightarrow$ every nbhd of A contains a point of $S - \{A\}$.

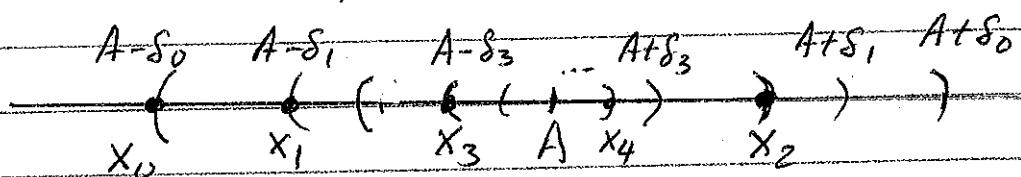
Proof: (\Rightarrow) Suppose A is an accumulation point of S . Let Q be any nbhd of A . Then Q contains infinitely many points of S . So Q will contain a point of S besides A , i.e., a point of $S - \{A\}$.

(\Leftarrow) Suppose every nbhd of A contains a point of $S - \{A\}$.

(13)

Let Q_0 be any nbhd of A . Then Q_0 contains a point $x_0 \in Q_0 \cap (S - \{A\})$. Let $\delta_0 = |A - x_0|$. Then $\delta_0 > 0$ since $x_0 \notin S - \{A\}$. Put

$Q_1 = (A - \delta_0, A + \delta_0) \cap Q_0$. Then Q_1 is a nbhd of A and $x_0 \notin Q_1$. Since Q_1 is a nbhd of A we can find a point $x_1 \in Q_1 \cap (S - \{A\})$. Let $\delta_1 = |A - x_1|$. Then $0 < \delta_1 < \delta_0$. Put $Q_2 = (A - \delta_1, A + \delta_2) \cap Q_0$. Then Q_2 is a nbhd of A & $x_0, x_1 \notin Q_2$. So we can find a point $x_2 \in Q_2 \cap (S - \{A\})$. If we continue in this manner we will get an infinite sequence of distinct points $\langle x_n \rangle_{n \in \mathbb{N}}$ which are all in $Q_0 \cap S$. Since this is true for any Q_0 , it follows that A is an accumulation point of S .



Theorem 14: (Bolzano-Weierstrass Theorem)

Every bounded infinite set of real numbers has at least one accumulation point.

Proof: See textbook.

Theorem 15: If the sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence then it is convergent.

Proof: See textbook.

END