

## (1)

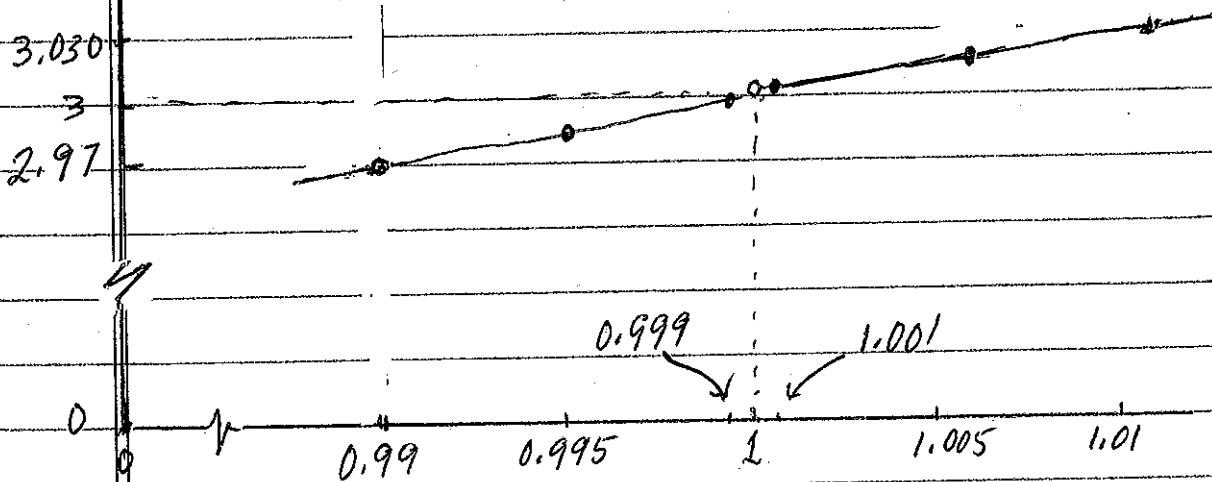
## Ch. 8 - Limits of Real functions

### §1. Limits of real partial functions.

Ex. 1 Consider the partial function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x) = (x^3 - 1)/(x - 1)$ . If we make a table of the values of  $f(x)$  when  $x$  is near 1, we get:

$x$	0.99	0.995	0.999	1	1.001	1.005	1.01
$f(x)$	2.970	2.985	2.9997	-	3.0003	3.015	3.030

From this we can see that  $f(x)$  approaches 3 as  $x$  approaches 1 (from both sides) & we write  $\lim_{x \rightarrow 1} f(x) = 1$  to indicate this



Def. Let  $a$  be a real number. We say that the set  $Q \subseteq \mathbb{R}$  is a punctured neighbourhood of  $a$  if  $(\exists \delta > 0) ((a - \delta, a + \delta) - \{a\} \subseteq Q)$ ,

Def. Let  $a \in \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a partial function which is defined in a punctured nbhd of  $a$ . We say that  $\lim_{x \rightarrow a} f(x) = L$  if  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta)(|f(x) - L| < \epsilon)$ .

(2)

Def

We say that  $\lim_{x \rightarrow a} f(x)$  exists if there exists a real number  $L \in \mathbb{R}$  such that  $\lim_{x \rightarrow a} f(x) = L$ .

Ex. 2 If  $a=3$  and  $f(x) = \frac{2x^2 - 18}{x - 3}$ , prove that  $\lim_{x \rightarrow a} f(x) = 12$

Sol: Put  $L=12$ , let  $\epsilon > 0$  be given. Choose  $\delta = \epsilon/2$

Then for all  $x \in \mathbb{R}$  with  $0 < |x - a| < \delta$  we have

$$\begin{aligned}|f(x) - L| &= \left| \frac{2x^2 - 18}{x - 3} - 12 \right| \\&= \left| \frac{2(x+3)(x-3)}{x-3} - 12 \right| \\&= |2(x+3) - 12| \quad \text{because } x-3 \neq 0 \\&= |2x - 6| = 2|x-3| \\&< 2\delta = \epsilon.\end{aligned}$$

So  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta)(|f(x) - L| < \epsilon)$ .

$\therefore \lim_{x \rightarrow a} f(x) = L$ , i.e.,  $\lim_{x \rightarrow 3} f(x) = 12$ .

Ex. 3 Prove that  $\lim_{x \rightarrow 2} x^2 = 4$ .

Sol. Put  $f(x) = x^2$ ,  $a=2$ , &  $L=4$ . Let  $\epsilon > 0$  be given. Choose  $\delta = \min\{1, \epsilon/5\}$ . Then for all  $x \in \mathbb{R}$  with  $0 < |x - a| < \delta$  we have

$$\begin{aligned}|f(x) - L| &= |x^2 - 4| = |(x+2)(x-2)| \quad \Rightarrow \text{since } |x-2| < \delta \leq 1 \\&= |x+2| \cdot |x-2| \quad \Rightarrow -1 < x-2 < 1 \\&< 5 \cdot |x-2| \quad \Rightarrow -1 < x < 3 \\&< 5 \cdot \delta \leq \epsilon \quad \Rightarrow 3 < x+2 < 5\end{aligned}$$

So  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta)(|f(x) - L| < \epsilon)$

$\therefore \lim_{x \rightarrow a} f(x) = L$ , So  $\lim_{x \rightarrow 2} x^2 = 4$ .

(3)

Ex.4 Prove that  $\lim_{x \rightarrow 2} x^3 = 8$ .

Sol. Let  $\epsilon > 0$  be given. Choose  $\delta = \min\{1, \epsilon/19\}$

Then for all  $x \in \mathbb{R}$  with  $0 < |x-2| < \delta$  we have

$$\begin{aligned} |x^3 - 8| &= |(x-2)(x^2 + 2x + 4)| && \Rightarrow |x-2| < 1 \\ &= |x^2 + 2x + 4| \cdot |x-2| && \Rightarrow -1 < x-2 < 1 \\ &< 19 \cdot |x-2| && \Rightarrow 1 < x < 3 \\ &< 19\delta && \Rightarrow 7 < x^2 + 2x + 4 < 19 \\ &\leq \epsilon. \end{aligned}$$

So  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-2| < \delta)(|x^3 - 8| < \epsilon)$ .

$$\therefore \lim_{x \rightarrow 2} x^3 = 8.$$

Ex.5 Prove that  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$ . (see Ex.1)

Sol. Let  $\epsilon > 0$  be given. Choose  $\delta = \min\{1, \epsilon/4\}$ .

Then for all  $x \in \mathbb{R}$  with  $0 < |x-1| < \delta$  we have

$$\begin{aligned} \left| \frac{x^3 - 1}{x - 1} - 3 \right| &= \left| \frac{(x-1)(x^2 + x + 1) - 3}{x-1} \right| \\ &= |x^2 + x + 1 - 3| \\ &= |x^2 + x - 2| && \Rightarrow |x-1| < 1 \\ &= |(x+2)(x-1)| && \Rightarrow -1 < x-1 < 1 \\ &= |x+2| \cdot |x-1| && \Rightarrow 2 < x+2 < 4 \\ &\leq 4 \cdot |x-1| \\ &< 4 \cdot \delta \leq \epsilon. \end{aligned}$$

$\therefore (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-1| < \delta)(\left| \frac{x^3 - 1}{x - 1} - 3 \right| < \epsilon)$ .

$$\therefore \lim_{x \rightarrow 1} [(x^3 - 1)/(x - 1)] = 3.$$

H.W. Prove that (a)  $\lim_{x \rightarrow -3} x^2 = 9$  (b)  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} = \frac{3}{2}$ ,

(4)

Prop. 1 If  $\lim_{x \rightarrow a} f(x) = L_1$  and  $\lim_{x \rightarrow a} f(x) = L_2$ , then  $L_1 = L_2$ .

Proof. Suppose  $L_1 \neq L_2$ . Let  $\varepsilon = |L_1 - L_2|/2$ . Since  $\lim_{x \rightarrow a} f(x) = L_1$ ,  $(\exists \delta_1 > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_1)(|f(x) - L_1| < \varepsilon)$ .

Also since  $\lim_{x \rightarrow a} f(x) = L_2$ ,

$(\exists \delta_2 > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_2)(|f(x) - L_2| < \varepsilon)$ . Let

$$\delta = \min\{\delta_1, \delta_2\}. \text{ Then}$$

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(a + \delta/2) + f(a + \delta/2) - L_2| \\ &\leq |L_1 - f(a + \delta/2)| + |f(a + \delta/2) - L_2| \\ &= |f(a + \delta/2) - L_1| + |f(a + \delta/2) - L_2| \\ &< \varepsilon + \varepsilon = 2\varepsilon = |L_1 - L_2| \end{aligned}$$

$\therefore |L_1 - L_2| < |L_1 - L_2|$  - a contradiction. Hence  $L_1 = L_2$ .

Def. Let  $a \in \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a partial function which is defined in a punctured nbhd of  $a$ . We say that  $f$  is locally bounded near  $a$  if  $(\exists \delta > 0)(\exists M > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta)(|f(x)| \leq M)$ .

Prop. 2 Suppose  $\lim_{x \rightarrow a} f(x)$  exists. Then  $f$  is locally bounded at  $a$ .

Proof. Suppose  $\lim_{x \rightarrow a} f(x)$  exists. Then  $\exists L \in \mathbb{R}$  such that  $\lim_{x \rightarrow a} f(x) = L$ .

Let  $\varepsilon = 1$ . Then  $(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta)(|f(x) - L| < 1)$ .

So for all  $x$  with  $0 < |x-a| < \delta$ ,  $-1 < f(x) - L < 1$

$$\text{i.e., } L-1 < f(x) < L+1.$$

Let  $M = \max\{|L-1|, |L+1|\}$ . Then for all  $x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ ,  $|f(x)| \leq M$ .

Hence  $(\exists \delta > 0)(\exists M > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta)(|f(x)| \leq M)$ .

So  $f$  is locally bounded near  $a$ .

Ex 6 Let  $f(x) = \frac{1}{x}$ . Then  $f$  is not locally bounded near 0.

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Def Let  $a \in \mathbb{R}$ . We say that  $Q \subseteq \mathbb{R}$  is a left-neighbourhood of  $a$  if  $(\exists \delta > 0)( (a-\delta, a) \subseteq Q )$ .

We say that  $Q \subseteq \mathbb{R}$  is a right-neighbourhood of  $a$  if  $(\exists \delta > 0)( (a, a+\delta) \subseteq Q )$ .

Def. (a) Let  $a \in \mathbb{R}$  &  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a partial function which is defined in a left-nbhd of  $a$ . We say that  $\lim_{x \rightarrow a^-} f(x) = L$  if  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < a-x < \delta)(|f(x)-L| < \varepsilon)$ .

(b) Let  $a \in \mathbb{R}$  &  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a partial function which is defined in a right-nbhd of  $a$ . We say that  $\lim_{x \rightarrow a^+} f(x) = L$  if  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < x-a < \delta)(|f(x)-L| < \varepsilon)$ .

Def. We say that  $Q \subseteq \mathbb{R}$  is a neighbourhood of  $+\infty$  if  $(\exists s > 0)( (s, \infty) \subseteq Q )$ .

We say that  $Q \subseteq \mathbb{R}$  is a neighbourhood of  $-\infty$  if  $(\exists s > 0)( (-\infty, -s) \subseteq Q )$ .

Def (a) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a partial function which is defined in a nbhd of  $+\infty$ . We say that  $\lim_{x \rightarrow +\infty} f(x) = L$  if  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } x > \delta)(|f(x)-L| < \varepsilon)$ .

(b) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a partial function which is defined in a nbhd of  $-\infty$ . We say that  $\lim_{x \rightarrow -\infty} f(x) = L$  if  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } x < -\delta)(|f(x)-L| < \varepsilon)$ .

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## §2. The Algebra of limits

Theorem 3 Let  $a \in \mathbb{R}$  and suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  are partial functions with  $\lim_{x \rightarrow a} f(x) = L_1$  &  $\lim_{x \rightarrow a} g(x) = L_2$ .

Then (a)  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot L_1$  for each  $c \in \mathbb{R}$ .

$$(b) \lim_{x \rightarrow a} [f(x) + g(x)] = L_1 + L_2$$

$$(c) \lim_{x \rightarrow a} [f(x) - g(x)] = L_1 - L_2.$$

Proof. (a) If  $c = 0$ , then  $c \cdot f(x) \equiv 0$ , and since  $\lim_{x \rightarrow a} (0) = 0$ , it follows that  $\lim_{x \rightarrow a} [0 \cdot f(x)] = 0 = 0 \cdot L_1$ . So suppose  $c \neq 0$ .

Let  $\epsilon > 0$  be given. Then  $\frac{\epsilon}{|c|} > 0$ . Since  $\lim_{x \rightarrow a} f(x) = L_1$

$(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta) (|f(x) - L_1| < \epsilon/|c|)$ .

So for all  $x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ , we have

$$\begin{aligned} |c \cdot f(x) - c \cdot L_1| &= |c \cdot (f(x) - L_1)| \\ &= |c| \cdot |f(x) - L_1| \\ &< |c| \cdot (\epsilon/|c|) = \epsilon. \end{aligned}$$

Hence  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta) (|c \cdot f(x) - c \cdot L_1| < \epsilon)$

$$\therefore \lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot L_1.$$

(b) Let  $\epsilon > 0$  be given. Then  $\epsilon/2 > 0$ . Since  $\lim_{x \rightarrow a} f(x) = L_1$ ,

$(\exists \delta_1 > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_1) (|f(x) - L_1| < \epsilon/2)$ . Also

since  $\lim_{x \rightarrow a} g(x) = L_2$ ,

$(\exists \delta_2 > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_2) (|g(x) - L_2| < \epsilon/2)$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then  $\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta$ ,

$$\begin{aligned} |f(x) + g(x) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\ &\leq |f(x) - L_1| + |g(x) - L_2| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

So  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta) (|f(x) + g(x) - (L_1 + L_2)| < \epsilon)$

$$\text{Hence } \lim_{x \rightarrow a} [f(x) + g(x)] = L_1 + L_2.$$

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$$\begin{aligned}
 (c) \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} [f(x) + (-1) \cdot g(x)] \\
 &= \lim_{x \rightarrow a} [f(x)] + \lim_{x \rightarrow a} [(-1) \cdot g(x)] \text{ by (a)} \\
 &= L_1 + (-1) \cdot L_2 = L_1 - L_2.
 \end{aligned}$$

Theorem 4 Let  $a \in \mathbb{R}$  and suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  &  $g: \mathbb{R} \rightarrow \mathbb{R}$  are partial functions with  $\lim_{x \rightarrow a} f(x) = L_1$  &  $\lim_{x \rightarrow a} g(x) = L_2$ . Then  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = L_1 \cdot L_2$

Proof: Let  $\varepsilon > 0$  be given. Since  $\lim_{x \rightarrow a} f(x)$  exists, it is locally bounded near  $a$ . So we can find a  $\delta_0 > 0$  &  $M > 0$  such that ( $\forall x \in \mathbb{R}$  with  $0 < |x-a| < \delta_0$ ) ( $|f(x)| \leq M$ ). Let  $\varepsilon' = \varepsilon / (M + |L_2|)$ . Then  $\varepsilon' > 0$ . Since  $\lim_{x \rightarrow a} f(x) = L_1$  &  $\lim_{x \rightarrow a} g(x) = L_2$ ,

$(\exists \delta_1 > 0)$  ( $\forall x \in \mathbb{R}$  with  $0 < |x-a| < \delta_1$ ) ( $|f(x) - L_1| < \varepsilon'$ ) and  $(\exists \delta_2 > 0)$  ( $\forall x \in \mathbb{R}$  with  $0 < |x-a| < \delta_2$ ) ( $|g(x) - L_2| < \varepsilon'$ ). Let  $\delta = \min \{\delta_0, \delta_1, \delta_2\}$ . Then  $\delta > 0$  &  $\forall x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ ,

$$\begin{aligned}
 |f(x) \cdot g(x) - L_1 \cdot L_2| &= |f(x) \cdot g(x) - f(x) \cdot L_2 + f(x) \cdot L_2 - L_1 \cdot L_2| \\
 &= |f(x) \cdot (g(x) - L_2) + L_2 (f(x) - L_1)| \\
 &\leq |f(x) \cdot (g(x) - L_2)| + |L_2 \cdot (f(x) - L_1)| \\
 &= |f(x)| \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
 &< M \cdot \varepsilon' + |L_2| \cdot \varepsilon' \\
 &= \varepsilon' (M + |L_2|) = \varepsilon.
 \end{aligned}$$

So ( $\forall \varepsilon > 0$ ) ( $\exists \delta > 0$ ) ( $\forall x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ ) ( $|f(x) \cdot g(x) - L_1 \cdot L_2| < \varepsilon$ )

Hence  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = L_1 \cdot L_2$ .

Lemma 5 Suppose  $\lim_{x \rightarrow a} g(x) = L$  and  $L \neq 0$ . Then we can find a  $\delta > 0$  such that  $\forall x \in \mathbb{R}$  with  $0 < |x-a| < \delta$   $|g(x)| \geq |L|/2$ .

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Proof

Let  $\varepsilon = |L|/2$ . Since  $L \neq 0$ ,  $\varepsilon > 0$ . Also since  $\lim_{x \rightarrow a} g(x) = L$ , we can find a  $\delta > 0$  such that for all  $x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ ,  $|g(x)-L| < \varepsilon$ .

So for all  $x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ ,

$$\begin{aligned}|g(x)| &= |L - (L - g(x))| \\&\geq |L| - |L - g(x)| \\&= |L| - |g(x) - L| \\&> |L| - (|L|/2) = |L|/2.\end{aligned}$$

$$\therefore (\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta) (|g(x)| \geq |L|/2)$$

Theorem 6 Suppose  $\lim_{x \rightarrow a} f(x) = L_1$  &  $\lim_{x \rightarrow a} g(x) = L_2$  and  $L_2 \neq 0$ . Then  $\lim_{x \rightarrow a} f(x)/g(x) = L_1/L_2$ .

Proof: Let  $\varepsilon > 0$  be given. Since  $\lim_{x \rightarrow a} g(x) = L_2$  and  $L_2 \neq 0$ , we can find a  $\delta_0 > 0$  such that for all  $x \in \mathbb{R}$  with  $0 < |x-a| < \delta_0$ ,  $|g(x)| \geq |L_2|/2$ . Let

$$\varepsilon' = \varepsilon \cdot |L_2|^2 / 2(|L_1| + |L_2|). \quad \text{Then } \varepsilon' > 0.$$

Since  $\lim_{x \rightarrow a} f(x) = L_1$  &  $\lim_{x \rightarrow a} g(x) = L_2$ ,  $(\exists \delta_1 > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_1)(|f(x) - L_1| < \varepsilon')$  and  $(\exists \delta_2 > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_2)(|g(x) - L_2| < \varepsilon')$ .

Let  $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ . Then  $\delta > 0$  and for all  $x \in \mathbb{R}$  with  $0 < |x-a| < \delta$ , we have

$$\begin{aligned}\left| \frac{f(x)}{g(x)} - \frac{L_1}{L_2} \right| &= \left| \frac{\{f(x), L_2 - L_1, g(x)\}}{\{g(x), L_2\}} \right| \\&= \frac{|f(x)L_2 - L_1L_2 + L_1L_2 - L_1g(x)|}{|g(x)L_2|} \\&= \frac{|(f(x) - L_1)L_2| + |L_1(L_2 - g(x))|}{|L_2|} \\&\leq 2\{|f(x) - L_1|, |L_2|\} + |L_1||g(x) - L_2| / |L_2|^2 \\&< 2(\varepsilon' \cdot |L_2| + \varepsilon' \cdot |L_1|) / |L_2|^2 = \varepsilon.\end{aligned}$$

So  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta)(|f(x)/g(x) - L_1/L_2| < \varepsilon)$ .

Hence  $\lim_{x \rightarrow a} f(x)/g(x) = L_1/L_2$ .

(9)

Def. Let  $f: R \rightarrow R$  be a partial function and suppose that  $f(x)$  is defined in a punctured nbhd of  $a$ . We say that  $f$  is well-behaved near  $a$  if  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in R \text{ with } 0 < |x-a|, |y-a| < \delta)(|f(x) - f(y)| < \varepsilon)$ .

Prop. 7 If  $\lim_{x \rightarrow a} f(x)$  exists, then  $f$  is well-behaved near  $a$ .

Proof: Suppose  $\lim_{x \rightarrow a} f(x)$  exists. Then we can find an  $L \in R$  such that  $\lim_{x \rightarrow a} f(x) = L$ . Let  $\varepsilon > 0$  be given.

Then  $\frac{\varepsilon}{2} > 0$ . Since  $\lim_{x \rightarrow a} f(x) = L$ , we can find a  $\delta > 0$  such that  $(\forall x \in R \text{ with } 0 < |x-a| < \delta)(|f(x) - L| < \frac{\varepsilon}{2})$ .

So for all  $x, y \in R$  with  $0 < |x-a|, |y-a| < \delta$  we have

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - L + L - f(y)| \\ &\leq |f(x) - L| + |L - f(y)| \\ &= |f(x) - L| + |f(y) - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So  $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in R \text{ with } 0 < |x-a|, |y-a| < \delta)(|f(x) - f(y)| < \varepsilon)$ . Hence  $f$  is well-behaved near  $a$ .

Prop. 8: Suppose  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a sequence which converges to  $a$  and  $f$  is well-behaved near  $a$ . Then  $\langle f(a_n) \rangle_{n \in \mathbb{N}}$  is a Cauchy sequence, and hence  $\langle f(a_n) \rangle_{n \in \mathbb{N}}$  converges to some real number  $L$ .

Theorem 9 If  $f$  is well-behaved near  $a$ , then  $\lim_{x \rightarrow a} f(x)$  exists.

The proofs of Prop. 8 & Theorem 9 are omitted because there is no time for it in this course.

END.