LINEAR ALGEBRA & ITS APPLICATIONS

Ch.4 Vector Spaces, subspaces, & their bases (24 pages)

1. Abstract vector spaces & subspaces
2. Linear combinations & linear independence
3. Bases & dimensions of vector spaces
4. The four fundamental subspaces of a matrix
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Ch.5 Linear Maps & their Matrix Representations (15 pages)

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Ch. 1 - Vectors & Systems of linear Equations

§1 Column vectors & row vectors over $\mathbb{R}$

Recall that $\mathbb{R}$ is the set of all real numbers. An n-tuple is a function f from $\{1, 2, 3, \ldots, n\}$ to $\mathbb{R}$. We will denote an n-tuple by $\langle f(1), f(2), \ldots, f(n)\rangle$. So an n-tuple is just a sequence of n real numbers.

Def. A column vector is an ordered pair of the form $\langle f, 0\rangle$ where f is an n-tuple. A row vector is an ordered pair of the form $\langle f, 1\rangle$.

Ex. 1 Let $f : \{1, 2\} \rightarrow \mathbb{R}$ be defined by $f(1) = 5$ & $f(2) = 3$. Then $\underset{\sim}{u} = \langle f, 0\rangle$ is a column vector & $\vec{v} = \langle f, 1\rangle$ is a row vector. We usually write $\underset{\sim}{u}$ as a column of numbers and $\vec{v}$ as a row of numbers. So $\underset{\sim}{u}$ & $\vec{v}$ will be depicted as shown below.

$$\underset{\sim}{u} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 5 & 3 \end{pmatrix}$$

Def. The transpose of a column vector $\underset{\sim}{u} = \langle f, 0\rangle$ is defined by $\underset{\sim}{u}^T = \langle f, 1\rangle$. The transpose of a row vector $\vec{v} = \langle g, 1\rangle$ is defined by $\vec{v}^T = \langle g, 0\rangle$. So the transpose of a column vector is a row vector and vice versa. Note also that $(\underset{\sim}{u}^T)^T = \underset{\sim}{u}$

Ex. 2 If $\underset{\sim}{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} -3 & 7 \end{pmatrix}$, then $\underset{\sim}{u}^T = \begin{pmatrix} 3 & -2 \end{pmatrix}$ & $\vec{v}^T = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.

We will denote the set of all n-tuple of real numbers by $\mathbb{R}^n$ and the set of all column vectors & row vectors obtained from $\mathbb{R}^n$ by $\mathbb{R}^n_{\downarrow}$ and $\mathbb{R}^n_{\rightarrow}$, respectively. When there is no room for confusion, we will drop the subscripts "$\downarrow$" and "$\rightarrow$". When the term <u>vector from $\mathbb{R}^n$</u> is used, we usually mean a column vector obtained from $\mathbb{R}^n$ because a column vector is considered more fundamental than a row vector. (Some textbooks refer to column vectors as simply vectors and refer to row vectors as co-vectors.)

<u>Def.</u> A <u>scalar</u> is just a real number. We define the <u>sum</u> of two vectors $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ & $\underset{\sim}{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ by $\underset{\sim}{u}+\underset{\sim}{v} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix}$.

We define the scalar product of α with $\underset{\sim}{u}$ by $\alpha \underset{\sim}{u} = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix}$

<u>Ex.3</u> Let $\underset{\sim}{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ & $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Then $\underset{\sim}{u}+\underset{\sim}{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ & $(-2)\underset{\sim}{u} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

<u>Def</u> The <u>zero vector</u> in $\mathbb{R}^n$ is defined by $\underset{\sim}{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. If $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ we define $(-\underset{\sim}{u})$ to be $\begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix}$.

<u>Prop 1</u>. Let $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ be vectors in $\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then

(a) $(\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w} = \underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w})$

(b) $\underset{\sim}{u}+\underset{\sim}{v} = \underset{\sim}{v}+\underset{\sim}{u}$

(c) $\underset{\sim}{u}+\underset{\sim}{0} = \underset{\sim}{u}$

(d) $\underset{\sim}{u}+(-\underset{\sim}{u}) = \underset{\sim}{0}$

(e) $\alpha(\underset{\sim}{u}+\underset{\sim}{v}) = \alpha\underset{\sim}{u}+\alpha\underset{\sim}{v}$

(f) $(\alpha+\beta)\underset{\sim}{u} = \alpha\underset{\sim}{u}+\beta\underset{\sim}{u}$

(g) $\alpha(\beta\underset{\sim}{u}) = (\alpha\beta)\underset{\sim}{u}$

(h) $1(\underset{\sim}{u}) = \underset{\sim}{u}$.

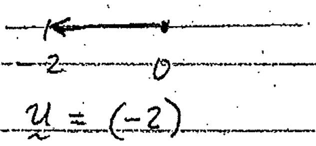
(i) $0(\underset{\sim}{u}) = \underset{\sim}{0}$

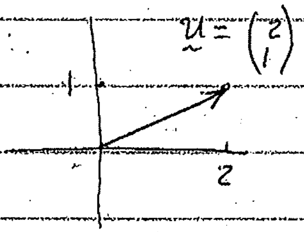
(j) $(-1)\underset{\sim}{u} = (-\underset{\sim}{u})$

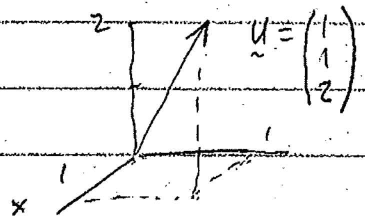
<u>Proof</u>: The results all follow from the definitions of vector addition, scalar multiplication, and the properties of $\mathbb{R}$.

Def. The real numbers $u_1, \ldots, u_n$ which are the constituents of a vector $\underset{\sim}{u}$ are called the components of $\underset{\sim}{u}$. The number of components of $\underset{\sim}{u}$ is called the dimension of $\underset{\sim}{u}$.

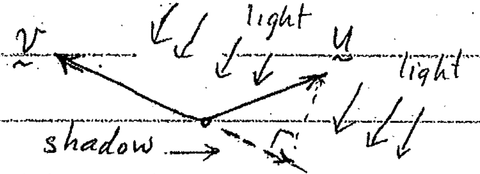
We can visualize vectors in 1, 2 or 3 dimensions as points on the real line $\mathbb{R}$, the plane $\mathbb{R}^2$, & the three-dimesional space $\mathbb{R}^3$. We often use a directed line segment from the origin to the point in question to get a physical representation of the vector.







Motivated by these physical representations, we can arrive at the concept of the length, $\|\underset{\sim}{u}\|$, of a vector. We can also get the concept of the length of the shadow of a vector $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$. This can lead us to the dot product $\underset{\sim}{u} . \underset{\sim}{v}$ as (length of the shadow of $\underset{\sim}{u}$ in the dir. of $\underset{\sim}{v}$)($\|\underset{\sim}{v}\|$)



Note the length of the shadow is negative.

Def. We define the length (or magnitude) of a vector $\underset{\sim}{u}$ in $\mathbb{R}^n$ by $\|\underset{\sim}{u}\| = \sqrt{(u_1^2 + u_2^2 + \cdots + u_n^2)}$. We define the dot product of $\underset{\sim}{u}$ & $\underset{\sim}{v}$ by $\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.

Note that $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^{\#}$ and $(- \cdot -) : \mathbb{R}^n \to \mathbb{R}$. Here $\mathbb{R}^{\#}$ = the set of non-negative real numbers. The dot product is also known as the inner product

Prop. 2 Let u, v, w be vectors in $\mathbb{R}^n$ & α be a scalar. Then

(a) $u \cdot u \geq 0$ with equality iff $u = 0$

(b) $u \cdot v = v \cdot u$ (symmetric property)

(c) $u \cdot (v + w) = (u \cdot v) + (u \cdot w)$

(d) $(\alpha u) \cdot v = \alpha (u \cdot v) = u \cdot (\alpha v)$

Proof: (a) $u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$. Also $u \cdot u = 0$ iff $u_1 = u_2 = \cdots = u_n = 0$ iff $u = 0$

(b)
$$\begin{aligned} u \cdot v &= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\ &= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n = v \cdot u. \end{aligned}$$

(c)
$$\begin{aligned} u \cdot (v + w) &= u_1 . (v_1 + w_1) + u_2 . (v_2 + w_2) + \cdots + u_n . (v_n + w_n) \\ &= (u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) + u_1 w_1 + u_2 w_2 + \cdots + u_n w_n \\ &= (u \cdot v) + u \cdot w. \end{aligned}$$

(d)
$$\begin{aligned} (\alpha u) \cdot v &= (\alpha u_1) v_1 + (\alpha u_2) v_2 + \cdots + (\alpha u_n) v_n \\ &= \alpha (u_1 v_1 + u_2 v_2 + \cdots + u_n) = \alpha (u \cdot v) \\ &= u_1 . (\alpha v_1) + u_2 . (\alpha v_2) + \cdots + u_n (\alpha v_n) = u \cdot (\alpha v). \end{aligned}$$

Prop. 3 Let u & v be vectors in $\mathbb{R}^n$ and α be a scalar. Then

(a) $\|u\| \geq 0$ with equality iff $u = 0$.

(b) $\|\alpha u\| = |\alpha| . \|u\|$ (Here $|\alpha|$ = absolute value of α).

(c) $\|u + v\| \leq \|u\| + \|v\|$ (triangular inequality).

Proof (a) $\|u\| = \sqrt{(u_1^2 + u_2^2 + \cdots + u_n^2)} = \sqrt{u \cdot u} \geq 0$. Also $\|u\| = 0$ iff $\sqrt{u \cdot u} = 0$ iff $u = 0$.

(b) $\|\alpha u\| = \sqrt{(\alpha u) \cdot (\alpha u)} = \sqrt{\alpha^2 (u \cdot u)} = |\alpha| \sqrt{u \cdot u} = |\alpha| \|u\|$.

(c) $\|u + v\|^2 = (u + v) \cdot (u + v) = u \cdot u + 2 u \cdot v + v \cdot v \leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2$

$\therefore \|u + v\|^2 \leq (\|u\| + \|v\|)^2$. So $\|u + v\| \leq \|u\| + \|v\|$. We will prove $|u \cdot v| \leq \|u\| . \|v\|$ (Cauchy-Swartz Inequality) later on.

§2 Systems of linear equations in 2 & 3 unknowns

Def. A linear equation in the unknowns $x_1, \dots, x_n$ is an equation of the form $c_1 x_1 + c_2 x_2 + \dots + c_n x_n = b$, where $b, c_1, \dots, c_n$ are constants from $\mathbb{R}$. The linear equation is non-trivial if at least one of the coefficients c_i is non-zero. It is homogeneous if $b = 0$. A system of linear equation is a sequence of linear equations. We will use a_{ij} to denote the coefficient of x_j in the i-th equation of the system. So a system of m equations in n unknowns can be written as:

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}$$

We can also write this system as a single vector equation

$$x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Def. A solution of the m by n system is any vector $\underset{\sim}{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ such that if we replace x_i by c_i then each of the m equations will become true.

The solution set of the system is the set of all vectors $\underset{\sim}{c}$ in $\mathbb{R}^n$ such $\underset{\sim}{c}$ is a solution of the system.

Let us look at all the solutions of a single linear equation

(*) $c_1 x_1 + c_2 x_2 = b_1$

in two unknowns x_1 and x_2. If the equation (*) is *non-trivial*, the set of all solutions of (*) will be a straight line in the plane $\mathbb{R}^2$. If (*) is both non-trivial & homogeneous, then the straight line will pass through the origin. If (*) is the trivial equation

$$0.x_1 + 0.x_2 = b_1$$

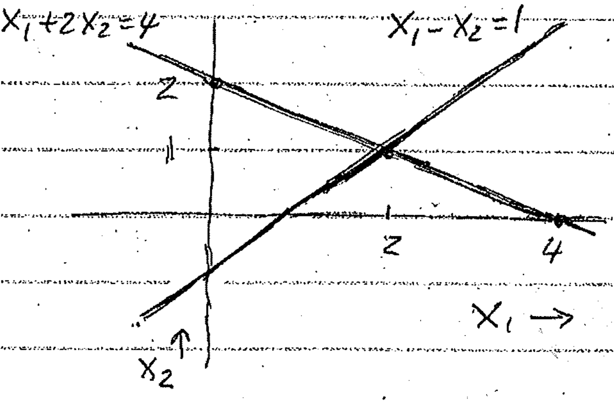
then (*) will have no solutions if $b_1 \neq 0$, and the solution set of (*) will be the whole plane $\mathbb{R}^2$ if $b_1 = 0$. If a trivial equation has no solution, we say that it is *inconsistent*. So a trivial, non-homogeneous equation is inconsistent. Now let us look at what solution sets are possible if we have a system of two non-trivial linear equations in two unknowns.

1 (a) We can get two straight lines which intersect in a single point.

Ex. 1(a) $x_1 + 2x_2 = 4$

$x_1 - x_2 = 1$

Solution set $= \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$



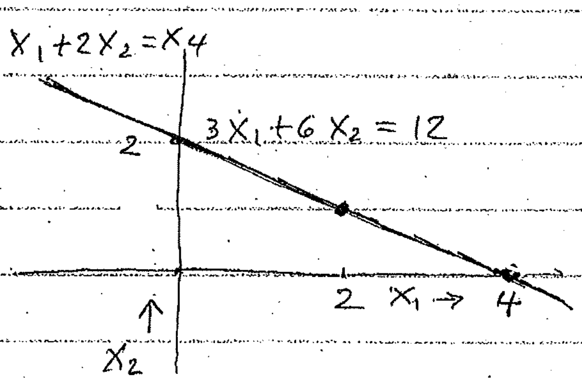
(b) We can get two straight lines which are coincident

Ex. 1(b) $x_1 + 2x_2 = 4$

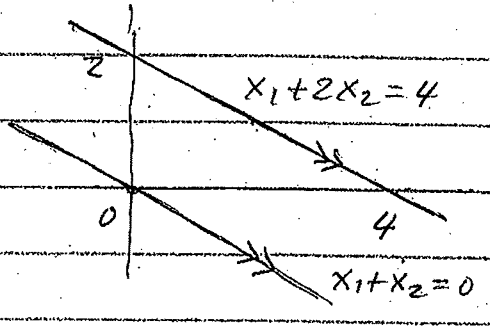
$3x_1 + 6x_2 = 12$

Solution set $= \left\{ \begin{pmatrix} 4-2\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$



(c) Finally we can get two non-coincident straight lines which are parallel.

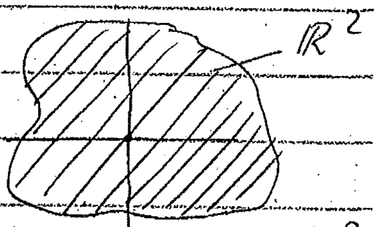


Ex.1(c) $x_1 + 2x_2 = 4$

$x_1 + 2x_2 = 0$

Solution set $= \emptyset$.

(d) If we allow trivial equations, there is one more type of solution set that we can get with two equations.



Ex.1(d) $0.x_1 + 0.x_2 = 0$

$0.x_1 + 0.x_2 = 0$

Solution set $= \left\{ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

$= \left\{ \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$

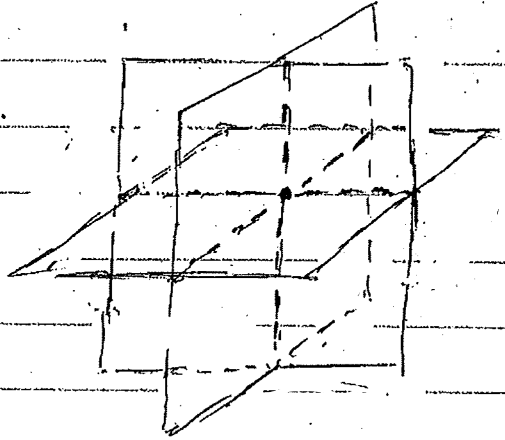
Note that if at least one of the equations is inconsistent then the solution set will be the empty set, $\emptyset$.

Let us now look at what solution sets are possible if we have a system of three non-trivial equations in 3 unknowns. The set of all solutions of a single linear equation

$$c_1x_1 + c_2x_2 + c_3x_3 = b_1$$

in 3 unknowns is always a plane in $\mathbb{R}^3$. If the equation is homogeneous, the plane will pass through the origin.

2 (a) Three planes intersecting in a point.



Ex. 2(a) $x_1 + 0.x_2 + 0x_3 = 1$

$0.x_1 + x_2 + 0x_3 = 3$

$0.x_1 + 0.x_2 - x_3 = 2$

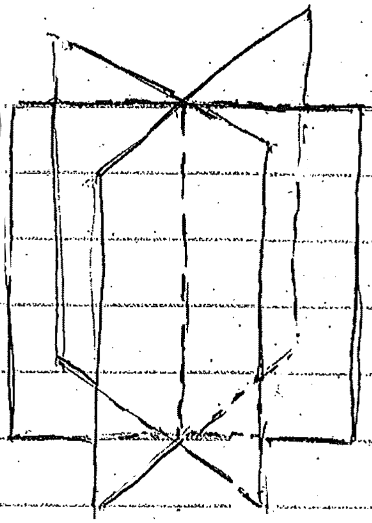
Solution set $= \left\{ \begin{pmatrix}1\\3\\-2\end{pmatrix} \right\}$.

2(b) Three planes (two of which may be coincident) intersecting in a line

Ex. 2b

$$\begin{aligned} x_1 + x_2 + 0.x_3 &= 1 \\ x_1 - x_2 + 0 \cdot x_3 &= 3 \\ 2x_1 + x_2 + 0 \cdot x_3 &= 0 \end{aligned}$$

Solutin set $= \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

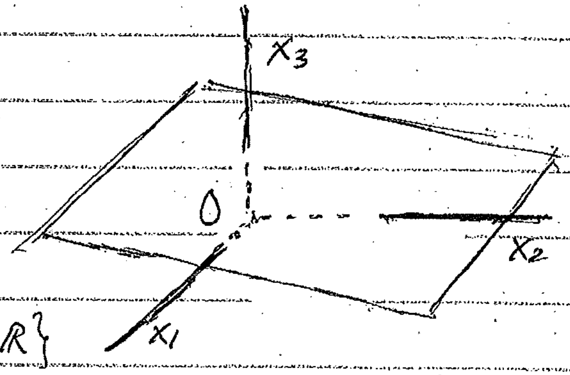


2(c) Three coincident planes.

Ex. 2(c)

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 + 2x_2 - 4x_3 &= 6 \\ -x_1 - x_2 + 2x_3 &= -3 \end{aligned}$$

Solution set $= \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

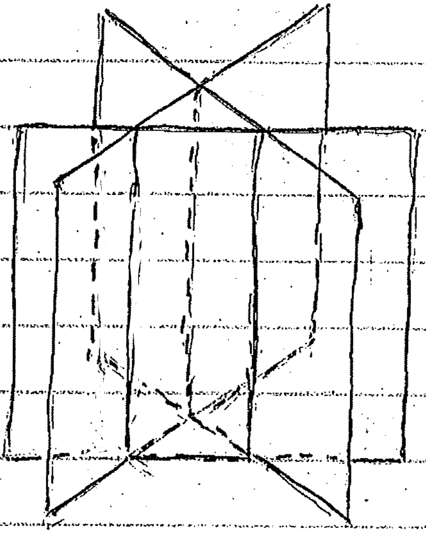


2(d) Three planes (two or three of which may be parallel) with no points in common.

Ex. 2(d)

$$\begin{aligned} x_1 + 0.x_2 + 0x_3 &= 2 \\ x_1 + x_2 + 0.x_3 &= 0 \\ x_1 - x_2 + 0.x_3 &= 0 \end{aligned}$$

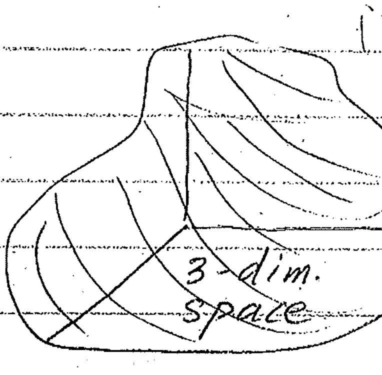
Solution set $= \emptyset$.



2(e) If we allow trivial equations, there is one more type of solution set that we can get with three equations

Ex. 2(e)

$$\begin{aligned} 0.x_1 + 0.x_2 + 0.x_3 &= 0 \\ 0.x_1 + 0.x_2 + 0.x_3 &= 0 \\ 0.x_1 + 0.x_2 + 0.x_3 &= 0 \end{aligned}$$



Solution set $= \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$

§3. Gaussian & Gauss-Jordan Elimination

Def. A system of linear equations is <u>consistent</u> if it has at least one solution. If it has no solutions, we say that the system is <u>inconsistent</u>. An <u>$m \times n$ system</u> is a system of m equations in n unknowns.

Def Two system of linear equations involving the same unknowns are <u>equivalent</u> if they have the same solution set.

There are three operations that can be used to obtain equivalent systems from a given system.

I interchange two equations

II replace an equation by a non-zero multiple of it

III replace an equation by itself plus a non-zero multiple of another equation.

Ex. 1 Find the solution set of the system $x_1 + 2x_2 + x_3 = 3,$

$$3x_1 - x_2 - 3x_3 = -1$$

$$2x_1 + 3x_2 + x_3 = 4$$

Sol.

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -x_2 - x_3 &= -2 \end{aligned}\right\} \begin{aligned} &E2 := E2 - 3E1 \\ &E3 := E3 - 2E1 \end{aligned}$$

$$\longrightarrow \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -x_2 - x_3 &= -2 \\ -7x_2 - 6x_3 &= -10 \end{aligned}\right\} \begin{aligned} &E2 := E3 \\ &E3 := E2 \end{aligned}$$

$$\longrightarrow \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -x_2 - x_3 &= -2 \\ x_3 &= 4 \end{aligned}\right\} E3 := E3 - 7E2$$

<u>triangular system</u>

Ex.1 So $x_3 = 4$. Substituting $x_3 = 4$ in the 2nd equation of the last system, we get $-x_2 - (4) = -2$. So $x_2 = -4 - (-2) = -2$. Finally substituting $x_3 = 4$ & $x_2 = -2$ in the 1st equation of the last system, we get $x_1 + 2(-2) + (4) = 3$. So $x_1 = 3$. Hence solution set is $\left\{ \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \right\}$.

This was a 3X3 system. The process of obtaining the equivalent system in <u>triangular form</u> is called <u>Gaussian elimination</u> (or <u>Chinese elimination</u>, because the same process was essentially done in the 3rd century BCE in China). The process of recovering the solution set from the triangular form system is called <u>back-substitution</u>.

Gaussian Elimination & back-substitution can be done on an mxn system. The resulting system we get is called an <u>echelon system</u>

<u>Ex. 2</u> Find the solution set of the system

$$\begin{aligned} x_1 + x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= 4 \\ -x_1 - x_2 - x_3 + x_4 &= 2 \end{aligned}$$

<u>Sol.</u>

$$\left.\begin{aligned} x_1 + x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= 4 \\ -x_1 - x_2 - x_3 + x_4 &= 2 \end{aligned}\right\} \rightarrow \left.\begin{aligned} x_1 + x_2 + x_3 + 2x_4 &= 1 \\ -2x_3 - 3x_4 &= 3 \\ 3x_4 &= 3 \end{aligned}\right\} \begin{aligned} &E2 := E2 - E1 \\ &E3 := E3 + E1 \end{aligned}$$

<u>echelon system</u>

Ex2 (contd.) So $3x_4 = 3$ which implies $x_4 = 1$. Substituting $x_4 = 1$ in the 2nd equation, we get $-2x_3 - 3(1) = 3$. So $2x_3 = -6$ and hence $x_3 = -3$. Substituting $x_4 = 1$ & $x_3 = -3$ in the 1st equation, we get $x_1 + x_2 + (-3) + 2(1) = 1$ So $x_1 = 2 - x_2$. If we put $x_2 = \alpha$ (an arbitrary real number we can write the solution set as

$$S = \left\{ \begin{pmatrix} 2-\alpha \\ \alpha \\ -3 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

We can continue the elimination process further to ensure that each unknown appears only once in the final system and so that the coefficient of the first unknown in each equation is a "1". This process is called <u>Gauss-Jordan</u> elimination (or <u>complete</u> elimination) and the final system we obtain is called the <u>reduced echelon system</u>.

Ex.3 Find the solution set of the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ -x_1 + x_2 - 2x_3 &= 3 \\ x_1 - x_2 + 4x_3 &= -5 \end{aligned}$$

Sol.

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ -x_1 + x_2 - 2x_3 &= 3 \\ x_1 - x_2 + 4x_3 &= -5 \end{aligned}\right\} \rightarrow \left.\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ -x_2 - x_3 &= -1 \\ x_2 + 3x_3 &= -1 \end{aligned}\right\} \begin{aligned} & \\ &E2 := E2 + E1 \\ &E3 := E3 - E1 \end{aligned}$$

$$\rightarrow \left.\begin{aligned} x_1 \qquad + 3x_3 &= -2 \\ -x_2 - x_3 &= -1 \\ 2x_3 &= -2 \end{aligned}\right\} \begin{aligned} &E1 := E1 - 2E2 \\ & \\ &E3 : E3 + E2 \end{aligned}$$

Ex.3 (continued)

$$\rightarrow \left.\begin{array}{rrr} X_1 & & +3X = -2\\ & X_2 & +X_3 = 1\\ & & X_3 = -1\end{array}\right\} \begin{array}{l} \\ E2:=(-1)E2\\ E3:=(1/2)E3\end{array}$$

reduced echelon system

$$\rightarrow \left.\begin{array}{rrr} X_1 & & = 1\\ & X_2 & = 2\\ & & X_3 = -1\end{array}\right\} \begin{array}{l} E1:=E1-3E3\\ E2:=E2-E2\\ \\ \end{array}$$

So solution set $= \{(1,2,-1)^T\}$.

Ex.4 Find the solution set of the system

$$\begin{array}{r} X_1+3X_2+2X_3=0\\ -X_1-3X_2+0X_3=-2\\ 2X_1+6X_2+X_3=3\end{array}.$$

Sol.

$$\left.\begin{array}{r} X_1+3X_2+2X_3=0\\ -X_1-3X_2+0X_3=-2\\ 2X_1+6X_2+X_3=3\end{array}\right\} \rightarrow \left.\begin{array}{r} X_1+3X_2+2X_3=0\\ 2X_3=-2\\ -3X_3=3\end{array}\right\} \begin{array}{l} \\ E2:=E2+E1\\ E3:=E3-2E1\end{array}$$

$$\rightarrow \left.\begin{array}{r} X_1+3X_2+2X_3=0\\ X_3=-1\\ -3X_3=3\end{array}\right\} E2:=(1/2)E2$$

free variable

reduced echelon form

$$\rightarrow \left.\begin{array}{r} X_1+3X_2 \quad = 2\\ X_3=-1\\ 0=0\end{array}\right\} \begin{array}{l} E1:=E1-2E2\\ \\ E3:=E3+3E2\end{array}$$

$\therefore X_3=-1$ & $X_1=2-3X_2$. Let the free variable $x_2=\alpha$.

Then solution set $= \left\{\begin{pmatrix}2-3\alpha\\ \alpha\\ -1\end{pmatrix} : \alpha\in\mathbb{R}\right\} = \left\{\begin{pmatrix}2\\0\\-1\end{pmatrix}+\alpha\begin{pmatrix}-3\\1\\0\end{pmatrix} : \alpha\in\mathbb{R}\right\}$.

§4. The coefficient matrix & augmented matrix

Consider the system:

$$\begin{aligned} x_1 + x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= 4 \\ -x_1 - x_2 - x_3 + x_4 &= 2 \end{aligned}$$

The coefficient matrix of this system is the rectangular array of the coefficients of the equations.

$A[i,j]$ = coefficient of x_j in the i-th equation.

The augmented matrix of this system is the array $A[i,j]$ augmented with the column of numbers on the right hand side of the equations.

Ex.1 $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ $\quad A_M = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & -1 & -1 & 4 \\ -1 & -1 & -1 & 1 & 2 \end{array}\right]$

coefficient matrix $\qquad$ augmented matrix

A matrix is often defined as a rectangular array of numbers, arranged into m rows and n columns. It is, however, better to think of a matrix as a row of n column vectors or as a column of m row vectors.

Def Let A and B be $m \times n$ matrices and α be a scalar.

We define the sum of the matrices A & B by

$$(A+B)[i,j] = A[i,j] + B[i,j].$$

We define the scalar product of α with A by

$$(\alpha A)[i,j] = \alpha . A[i,j]$$

We say that $A = B$ if $A[i,j] = B[i,j]$ for each i & j.

Ex.2 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & -2 \\ 1 & -2 & -3 \end{bmatrix}$

Then $A+B = \begin{bmatrix} 1-2 & 2+1 & 3-2 \\ 4+1 & 5-2 & 6-3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 5 & 3 & 3 \end{bmatrix}$

and $(-3)B = \begin{bmatrix} (-3)(-2) & (-3)(1) & (-3)(-2) \\ (-3)(1) & (-3)(-2) & (-3)(-3) \end{bmatrix} = \begin{bmatrix} 6 & -3 & 6 \\ -3 & 6 & 9 \end{bmatrix}$

Def We define zero matrix $O_{m,n}$ by $O_{m,n}[ij] = 0$. If A is an $m \times n$ matrix, we define $(-A)$ to be the matrix defined by $(-A)[ij] = -A[ij]$. We will denote the set of all $m \times n$ matrices by $\mathbb{R}^{m \times n}$

Ex.3 Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$. Then $-A = \begin{bmatrix} -1 & 1 & -2 \\ -2 & -3 & 1 \end{bmatrix}$

Also $O_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Prop. 4 : Let $A, B, C \in \mathbb{R}^{m \times n}$, $O_{m \times n} = O$ and $\alpha, \beta \in \mathbb{R}$

(a) $(A+B)+C = A+(B+C)$ (e) $\alpha(A+B) = (\alpha A)+(\alpha B)$
(b) $A+B = B+A$ (f) $(\alpha+\beta)A = (\alpha A)+(\beta A)$.
(c) $A+O = A$ (g) $\alpha(\beta A) = (\alpha\beta)A$
(d) $A+(-A) = O$ (h) $1(A) = A$
(i) $0(A) = O$ (j) $(-1)A = (-A)$.

Proof: These results all follow from the definition of matrix addition & scalar multiplication just as in Prop. 1. (Note the similarity between Prop. 1 & Prop. 4). Do for Home work

An $m \times n$ matrix can be thought of as a column vector, of dimensions $m.n$, which is cut up into n pieces each of dimension m. So it is not surprising that Prop. 4 is so similar to Prop. 1. We can even define an inner product of two $m \times n$ matrices as follows.

Def. Let A & B be $m \times n$ matrices. We define $\langle A; B\rangle$ by $\langle A, B\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A[i,j].B[i,j]$.

Unfortunately this inner product is not very useful even though it has all the nice properties as in Prop. 3. In the next chapter we will define the matrix product and this will turn out to be the most important operation on matrices. Let us now go back to system of linear equations.

Def. 1. A matrix A is in row echelon form if the first non-zero entry of any row is to the right of the first non-zero entry of the row above it.

2. The first non-zero term in each row of a matrix in row echelon form is called the leading term

3. A matrix is in row echelon form with leading 1's if the leading term in each row is a "1".

4. A matrix is in reduced row echelon form if it is in row echelon form with leading 1's and all the other entries in the columns with leading 1's are zeros.

When we perform the three operations, that are used in Gaussian & Gauss-Jordan elimination, on the augmented matrix of a linear system we call them row operations.

Type I : interchange row i & row j

Type II : replace row i by α(row i) where $\alpha \neq 0$.

Type III : replace row i by row i + α(row j) with $\alpha \neq 0$.

Ex.1 Find the solution set of the system below by using row operations on the augmented matrix.

$$\begin{aligned} x_1 + x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= 4 \\ 2x_1 + 2x_2 + 0x_3 - x_4 &= 3 \\ -x_1 - x_2 - x_3 + x_4 &= 2 \end{aligned}$$

Sol.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & -1 & -1 & 4 \\ 2 & 2 & 0 & -1 & 3 \\ -1 & -1 & -1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & -2 & -3 & 3 \\ 0 & 0 & -2 & -5 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{l} \\ R2 := R2 - R1 \\ R3 := R3 - 2R1 \\ R4 := R4 + R1 \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{A} \quad \underset{\sim}{b}$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3/2 & -3/2 \\ 0 & 0 & 1 & 5/2 & -1/2 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{l} \\ R2 := (-1/2)R2 \\ R3 := (-1/2)R3 \\ \\ \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1/2 & 5/2 \\ 0 & 0 & 1 & 3/2 & -3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{l} R1 := R1 - R2 \\ \\ R3 := R3 - R2 \\ \\ \end{array}$$

Ex. 1 (continued)

reduced row echelon form, A_R

$$\rightarrow \left[\underbrace{\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}}_{A_R}\right|\left.\begin{array}{c} 2 \\ -3 \\ 1 \\ 0 \end{array}\right] \begin{array}{l} R1:=R1-\frac{1}{2}R3 \\ R2:=R2-\frac{3}{2}R3 \\ \\ R4:=R4-3R3 \end{array}$$

delete the row of zeros at the bottom

standard row echelon form, A_S

$$\rightarrow \left[\underbrace{\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}}_{A_s}\right|\left.\underbrace{\begin{array}{c} 2 \\ 0 \\ -3 \\ 1 \end{array}}_{\underset{\sim}{b_s}}\right]$$

insert rows of zeros so that all the leading 1's are in the diagonal & A_s is square.

The solution set in standard form is obtained from $\underset{\sim}{b_s}$ & the non-zero columns of $(I-A_s)$. We know

$$I-A_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So solution set $= \left\{ \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \alpha \in \mathbb{R} \right\}$.

Ex.2 Find the solution set, in standard form, of the system

$$x_1 + 2x_2 - x_3 + 3x_4 = 1$$
$$2x_1 + 4x_2 - x_3 + 4x_4 = 5$$

Sol.

$$\left[\underbrace{\begin{array}{cccc} 1 & 2 & -1 & 3 \\ 2 & 4 & -1 & 4 \end{array}}_{A}\right|\left.\underbrace{\begin{array}{c} 1 \\ 5 \end{array}}_{\underset{\sim}{b}}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 3 \end{array}\right] \quad R2:=R2-2R1$$

reduced row echelon form

$$\rightarrow \left[\underbrace{\begin{array}{cccc} 1 & 2 & 0 & +1 \\ 0 & 0 & 1 & -2 \end{array}}_{A_R}\right|\left.\begin{array}{c} 4 \\ 3 \end{array}\right] \quad R1:=R1+R2$$

<u>Ex. 2</u>

standard row echelon form

$$\longrightarrow \left[\underbrace{\begin{matrix} \underline{1} & 2 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{matrix}}_{A_s} \middle| \underbrace{\begin{matrix} 4 \\ 0 \\ 3 \\ 0 \end{matrix}}_{\underset{\sim}{b}_s} \right] \begin{matrix} \\ \leftarrow \text{row of 0's} \\ \\ \leftarrow \text{row of 0's} \end{matrix}$$

$$I - A_s = \begin{bmatrix} 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & +2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{Solution set} = \left\{ \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ +2 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

<u>Ex 3.</u> Find the solution set of the system

$$\begin{aligned} 0.X_1 + X_2 - X_3 &= 1 \\ X_1 + X_2 - 3X_3 &= 3 \\ X_1 + 2X_2 - 4X_3 &= 0 \end{aligned}$$

<u>Sol.</u>

$$\left[\underbrace{\begin{matrix} 0 & 1 & -1 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 \\ 3 \\ 0 \end{matrix}}_{\underset{\sim}{b}} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -4 & 0 \end{array}\right] \begin{matrix} R1 := R2 \\ R2 := R1 \\ \\ \end{matrix}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -3 \end{array}\right] \begin{matrix} \\ \\ R3 := R3 - R1 \end{matrix}$$

$$\longrightarrow \left[\underbrace{\begin{matrix} \underline{1} & 0 & -2 \\ 0 & \underline{1} & -1 \\ 0 & 0 & 0 \end{matrix}}_{A_R} \middle| \underbrace{\begin{matrix} 2 \\ 1 \\ -4 \end{matrix}}_{\underset{\sim}{b}_R} \right] \begin{matrix} R1 := R1 - R2 \\ \\ R3 := R3 - R2 \end{matrix}$$

Since the third equation of the final system we got says $0 = -4$, the system is inconsistent because it contains an inconsistent equation, namely $0 = -4$. Hence solution set $= \emptyset$. (There is no need to look at $I - A_s$ & $\underset{\sim}{b}_s$.)