Ch.2 – Matrix Algebra & Special Matrices

§1. Matrix multiplication and invertible matrices

Def. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the matrix product AB of A and B to be the $m \times p$ matrix defined by

$$(AB)[i,j] = \sum_{k=1}^{n} A[i,k] \cdot B[k,j].$$

Recall we can view an $m \times n$ matrix A as a column of m row vectors or as a row of n column vectors. We will use $\vec{x}$ to denote row vectors & $\underset{\sim}{x}$ to denote column vectors. So

$$A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix} \quad \text{and} \quad A = [\underset{\sim}{a_1}, \underset{\sim}{a_2}, \cdots, \underset{\sim}{a_n}]$$

Thus $(AB)[i,j] = (\vec{a}_i)^T \circ \underset{\sim}{b_j} = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} \circ \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$, where "$\circ$" is the inner product of vectors.

Ex.1 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} (1\ \ 2)^T \circ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & (1\ \ 2)^T \circ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\ (3\ \ 4)^T \circ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & (3\ \ 4) \circ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \circ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \circ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \circ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 \\ 4 \end{pmatrix} \circ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+2(0) & 1(1)+2(-3) \\ 3(2)+4(0) & 3(1)+4(-3) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 6 & -9 \end{bmatrix}.$$

We need A to be $m \times n$ and B to be $n \times p$ so that the dimensions of the rows of A will match the dimensions of the columns of B so that $(\vec{a}_i)^T \cdot (\underset{\sim}{b}_j)$ will be properly defined. There is another way of looking at the matrix product.

Def. Let $\underset{\sim}{u}$ be any column vector of dimension m and $\vec{v}$ be any row vector of dimension p. We define the <u>outer product</u> of $\underset{\sim}{u}$ and $\vec{v}$ to be the $m \times p$ matrix defined by

$$\underset{\sim}{u} * \vec{v} = \begin{bmatrix} u_1 \vec{v} \\ \vdots \\ u_m \vec{v} \end{bmatrix}$$

Note that $\underset{\sim}{u} * \vec{v} = [v_1 \underset{\sim}{u}, \; v_2 \underset{\sim}{u}, \; \dots, \; v_p \underset{\sim}{u}]$ also

A <u>simple matrix</u> is any matrix C that can be written in the form $\underset{\sim}{u} * \vec{v}$ for some $\underset{\sim}{u}$ & $\vec{v}$.

Ex.2 Let $\underset{\sim}{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\vec{v} = (3 \;\; 4)$ Then

$$\underset{\sim}{u} * \vec{v} = \begin{bmatrix} -1(3 \;\; 4) \\ 2(3 \;\; 4) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 6 & 8 \end{bmatrix}$$

Also $\underset{\sim}{u} * \vec{v} = \begin{bmatrix} 3\begin{pmatrix} -1 \\ 2 \end{pmatrix} & 4\begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 6 & 8 \end{bmatrix}$

We can express the matrix product of the $m \times n$ matrix A with the $n \times p$ matrix B as a sum of simple matrices.

$$AB = \sum_{k=1}^{n} \underset{\sim}{a}_k * \vec{b}_k$$

Ex.3 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$. Then

$$\begin{aligned} AB &= (\underset{\sim}{a_1} * \vec{b}_1) + (\underset{\sim}{a_2} * \vec{b}_2) \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} * (2 \;\; 1) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} * (0 \;\; -3) \\ &= \begin{bmatrix} 1(2 \;\; 1) \\ 3(2 \;\; 1) \end{bmatrix} + \begin{bmatrix} 2(0 \;\; -3) \\ 4(0 \;\; -3) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 0 & -12 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 6 & -9 \end{bmatrix} \end{aligned}$$

This is another reason why we need the number of columns of A to be the same as the number of rows of B, because $AB = (a_1 * \vec{b}_1) + (\underset{\sim}{a_2} * \vec{b}_2) + \cdots + (\underset{\sim}{a_n} * \vec{b}_n)$. Anyway, we don't need all of this to prove the following results. All we need is the original definition, $(AB)[i,j] = \sum_{k=1}^{n} A[i,k] \cdot B[k,j]$, that was obtained by the mathematician Cayley from his study of linear transformations.

Def. The matrix A is compatible with B if the number of columns of A is the same as the number of rows of B.

Prop. 1 Let A, B & C be matrices for which the following operations are possible & suppose A is $m \times n$. Then

(a) $A(B+C) = AB + AC$ (d) $(AB)C = A(BC)$

(b) $(A+B)C = AC + BC$ (e) $A\,O_n = O_{m,n} = O_m A$

(c) $\alpha(AB) = (\alpha A)B = A(\alpha B)$ (f) $A I_n = A = I_m A$

Here $O_n = O_{n,n}$ & I_n is the $n \times n$ matrix with
$$I_n[i,j] = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Proof:

(a) $$\underset{m\times n}{A}\,(\underset{n\times p}{B} + \underset{n\times p}{C})\,[i,j] = \sum_{k=1}^{n} A[i,k].(B+C)[k,j]$$
$$= \sum_{k=1}^{n} A[i,k].\left(B[k,j] + C[k,j]\right)$$
$$= \sum_{k=1}^{n} A[i,k].B[k,j] + \sum_{k=1}^{n} A[i,k].C[k,j]$$
$$= (AB)[i,j] + (AC)[i,j]$$
$\therefore\ A(B+C) = (AB) + (AC)$.

(b) Do for H.W.

(c) $$\alpha(\underset{m\times n}{A}\ \underset{n\times p}{B}) = \alpha.\sum_{k=1}^{n} A[i,k].B[k,j] = \sum_{k=1}^{n}(\alpha A[i,k]).B[k,j]$$
$$= \sum_{k=1}^{n}(\alpha A)[i,k].B[k,j] = \{(\alpha A)B\}[i,j]$$
$\therefore\ \alpha(AB) = (\alpha A)(B)$.

Prove $(\alpha A)(B) = A(\alpha B)$ for H.W.

(d) $$\{(\underset{m\times n}{A}\ \underset{n\times p}{B})\underset{p\times q}{C}\}[i,j] = \sum_{l=1}^{p}(AB)[i,l].C[l,j]$$
$$= \sum_{l=1}^{p}\left\{\sum_{k=1}^{n} A[i,k].B[k,l]\right\}.C[l,j]$$
$$= \sum_{l=1}^{p}\sum_{k=1}^{n} A[i,k].B[k,l].C[l,j]$$
$$= \sum_{k=1}^{n}\sum_{l=1}^{p} A[i,k].B[k,l].C[l,j]$$
$$= \sum_{k=1}^{n} A[i,k].\sum_{l=1}^{p} B[k,l].C[l,j]$$
$$= \sum_{k=1}^{n} A[i,k].(BC)[k,j] = \{A(BC)\}[i,j]$$
$\therefore\ (AB)C = A(BC)$.

(e) $\underset{m\times n\ n\times n}{(A\,O_n)}[i,j] = \sum_{k=1}^{n} A[i,k].O_n[k,j] = \sum_{k=1}^{n} A[i,k].0 = 0$

$\underset{m\times m\ m\times n}{(O_m A)}[i,j] = \sum_{k=1}^{m} O_m[i,k].A[k,j] = \sum_{k=1}^{m} 0.A[k,j] = 0.$

So $A\,O_n = O_{m,n} = O_m A$.

(f) $\underset{m\times n\ n\times n}{(A\,I_n)}[i,j] = \sum_{k=1}^{n} A[i,k].I_n[k,j]$

$= A[i,1].0 + \cdots + A[i,j].1 + \cdots + A[i,n].0 = A[i,j]$

$\underset{m\times m\ m\times n}{(I_m A)}[i,j] = \sum_{k=1}^{m} I_m[i,k].A[k,j]$

$= 0.A[1,j] + \cdots + 1.A[i,j] + \cdots + 0.A[n,j] = A[i,j]$

$\therefore\ A\,I_n = A = I_m A.$

<u>Def.</u> Let A be an $m\times n$ matrix. We define the <u>transpose</u> A^T of A to be the $n\times m$ matrix defined by

$(A^T)[i,j] = A[j,i].$

If we write $A = [\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n]$, Then $A^T = \begin{bmatrix} \underset{\sim}{a}_1^T \\ \vdots \\ \underset{\sim}{a}_n^T \end{bmatrix}$

Ex. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Then $A^T = \begin{bmatrix} \binom{1}{4}^T \\ \binom{2}{5}^T \\ \binom{3}{6}^T \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

<u>Prop. 2</u> Let A & B be matrices for which the following operations are possible & suppose A is $m\times n$. Then

(a) $(A+B)^T = A^T + B^T$ (c) $(A^T)^T = A$.

(b) $(\alpha A)^T = \alpha(A^T)$ (d) $(AB)^T = B^T A^T$.

<u>Proof:</u> (a) $(A+B)^T[i,j] = (A+B)[j,i] = A[j,i] + B[j,i]$

$= A^T[i,j] + B^T[i,j] = (A^T+B^T)[i,j]$

$\therefore\ (A+B)^T = A^T + B^T.$

(b) $(\alpha A)^T[i,j] = (\alpha A)[j,i] = \alpha . A[j,i] = \alpha . A^T[i,j] = (\alpha A^T)[i,j]$.
So $(\alpha A)^T = \alpha . A^T$.

(c) $(A^T)^T[i,j] = (A^T)[j,i] = A[i,j]$. $\therefore (A^T)^T = A$.

(d) $$(\underset{m\times n}{A}\underset{n\times p}{B})^T[i,j] = (AB)[j,i] = \sum_{k=1}^{n} A[j,k] . B[k,i]$$
$$= \sum_{k=1}^{n} A^T[k,j] . B^T[i,k] = \sum_{k=1}^{n} B^T[i,k] . A^T[k,j]$$
$$= (B^TA^T)[i,j].$$
$\therefore (AB)^T = B^TA^T$.

<u>Def.</u> Let A be an $n\times n$ matrix. We say that A is <u>invertible</u> if we can find an $n\times n$ matrix B such that $AB = I_n = BA$. The matrix B is called an <u>inverse</u> of A.

<u>Ex.1</u> Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$. Choose $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

& $BA = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So A is invertible. & B is an inverse of A.

<u>Prop.3</u>. An $n\times n$ matrix A can have at most one inverse.

<u>Proof.</u> Suppose B & C are both inverses of A. Then $AB = I_n = BA$ and $AC = I_n = CA$. So

$$C = CI_n = C(AB) \quad \text{because } AB = I_n$$
$$= (CA)B \quad \text{by the associative law}$$
$$= I_nB = B \quad \text{because } I_n = CA.$$

So $C = B$ and hence A can have at most one inverse.

Notation: We will denote the unique inverse of an $n \times n$ matrix A by A^{-1}, when it exists.

Prop. 4 Let A & B be invertible $n \times n$ matrices & $\alpha \neq 0$. Then I_n, A^T, A^{-1}, AB, & αA are all invertible and

(a) $I_n^{-1} = I_n$

(b) $(A^{-1})^{-1} = A$

(c) $(A^T)^{-1} = (A^{-1})^T$

(d) $(AB)^{-1} = B^{-1}A^{-1}$

(e) $(\alpha A)^{-1} = (\alpha^{-1})A^{-1}$

Proof: (a) $I_n(I_n) = I_n = (I_n)I_n$. So I_n is invertible and $(I_n)^{-1} = I_n$.

(b) Since A is invertible $A(A^{-1}) = I_n = (A^{-1})A$. Now $(A^{-1})A = I_n = A(A^{-1})$. So A^{-1} is invertible & $(A^{-1})^{-1} = A$.

(c) Since A is invertible, $A(A^{-1}) = I_n = (A^{-1})A$. So

$$(A^T)[(A^{-1})^T] = (A^{-1}A)^T = (I_n)^T = I_n. \text{ Also}$$
$$[(A^{-1})^T](A^T) = [A(A^{-1})]^T = I_n^T = I_n.$$

So A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

(d) Since A & B are invertible $AA^{-1} = I_n = A^{-1}A$ & $BB^{-1} = I_n = B^{-1}B$. So

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

& $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.

Hence (AB) is invertible & $(AB)^{-1} = B^{-1}A^{-1}$.

(e) Since A is invertible, $AA^{-1} = I_n = A^{-1}A$. So

$$(\alpha A)(\alpha^{-1}A^{-1}) = (\alpha\alpha^{-1})(AA^{-1}) = 1(I_n) = I_n$$
$$(\alpha^{-1}A^{-1})(\alpha A) = (\alpha^{-1}\alpha)(A^{-1}A) = 1(I_n) = I_n$$

So (αA) is invertible & $(\alpha A)^{-1} = (\alpha^{-1})A^{-1}$.

§2. Elementary matrices & computation of A^{-1}

Def An *elementary matrix* E_i of type i is any matrix that can be obtained from the identity matrix I_n by performing exactly one type i row operation, for $i = 1, 2 \& 3$.

Ex. 1 (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1$ (type 1 elem. matrix) $\begin{pmatrix} R2 := R3 \\ R3 := R2 \end{pmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$ (type 2) $[R2 := (3)R2]$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = E_3$ (type 3) $[R2 := R2 + 2R3]$

Ex. 2 (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 3 (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{31} & a_{22}+2a_{32} & a_{23}+2a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Prop. 5 Let E be an $n \times n$ elementary matrix and A be any $n \times n$ matrix. Then EA is the matrix obtained from A by using the same row operation that was used to obtain E from I_n.

Prop 6: An elementary matrix of type i is invertible and its inverse is an elementary matrix of type i (for $i = 1, 2 \& 3$).

Proof: (a) Let E_1 be an elementary matrix of type 1 obtained by switching row i & row j of I_n. Then by switching row i & row j in E_1, we will get back I_n. So $E_1 E_1 = I_n = E_1 E_1$. So E_1 is invertible and $E_1^{-1} = E_1$.

(b) Let E_α be an elementary matrix of type 2 obtained by replacing row i of I_n by α(row i) of I_n. Then replacing row i of E_α by (α^{-1})(row i), we will get back I_n. So $E_{\alpha^{-1}} E_\alpha = I_n$. Similarly $E_\alpha E_{\alpha^{-1}} = I_n$. So E_α is invertible and $(E_\alpha)^{-1} = E_{\alpha^{-1}}$.

(c) Let $E_{i+\alpha j}$ be an elementary matrix of type 3 obtained by replacing row i of I_n by (row i) + α(row j) of I_n. Then replacing row i of $E_{i+\alpha j}$ by (row i) $-$ α(row j) of $E_{i+\alpha i}$ we will get back I_n. So $E_{i-\alpha j} E_{i+\alpha j} = I_n$. Similarly, $E_{i+\alpha j} E_{i-\alpha j} = I_n$. So $E_{i+\alpha j}$ is invertible and $(E_{i+\alpha j})^{-1} = E_{i-\alpha j}$.

Prop. 7: Let $B = E_k E_{k-1} \cdots E_2 E_1$ be a product of elementary $n \times n$ matrices. Then B is invertible and $B^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$.

Proof: The proof is by induction on k.

Basis: If $k=2$, then the result is a direct consequence of Prop. 4 (d). So the result true for $k=2$.

Ind. step: Suppose the result is true for $k-1$ with $k \geq 3$. Then $E_{k-1} \cdots E_2 E_1$ is invertible and $(E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. Now let E_k be any elementary matrix. Then $B = (E_k)(E_{k-1} \cdots E_2 E_1)$. Since E_k & $(E_{k-1} \cdots E_2 E_1)$ are both invertible, it follows that $B = E_k(E_{k-1} \cdots E_2 E_1)$ is invertible and $B^{-1} = (E_{k-1} \cdots E_2 E_1)^{-1} E_k^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$. So if the result is true for $k-1$, it will be true for k.

Concl: Hence by the Principle of Mathematical Induction, it follows that the result is true for all k.

Theorem 8: Let A be an $n \times n$ matrix. If we can perform row operations and transform $[A / I_n]$ into $[I_n / B]$, then A is invertible and $A^{-1} = B$.

Proof: Suppose we can transform $[A / I_n]$ into $[I_n / B]$ by using row operations. Then we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k E_{k-1} \cdots E_2 E_1 A = I_n$. So

$$(E_k E_{k-1} \cdots E_1)[A / I_n] = [E_k E_{k-1} \cdots E_1 A / E_k E_{k-1} \cdots E_1 I_n]$$
$$= [I_n / E_k E_{k-1} \cdots E_1] = [I_n / B]$$

So $B = E_k E_{k-1} \cdots E_1$. Hence B is invertible. Now

$$BA = (E_k E_{k-1} \cdots E_1) A = I_n.$$

Also $AB = I_n(AB) = (B^{-1}B)(AB) = B^{-1}(BA)B$
$= B^{-1} I_n B = B^{-1} B = I_n$

So $AB = I_n = BA$. Hence A is invertible & $A^{-1} = B$.

Ex.4 Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 2 & 0 & 1 \end{bmatrix}$

Sol. $\left[\underbrace{\begin{matrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 2 & 0 & 1 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right] \to \left[\begin{matrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & -4 & -7 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{matrix}\right]$ $\begin{matrix} R2 := R2 - R1 \\ R3 := R3 - 2R1 \end{matrix}$

$\to \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ -6 & 4 & 1 \end{matrix}\right]$ $\begin{matrix} R1 := R1 - 2R2 \\ \\ R3 := R3 + 4R2 \end{matrix}$

$\to \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 3 & -2 & 0 \\ 11 & -7 & -2 \\ -6 & 4 & 1 \end{matrix}}_{A^{-1}}\right]$ $R2 := R2 - 2R3$

Check: $AA^{-1} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ 11 & -7 & -2 \\ -6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let us now return to systems of linear equations. Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can write this system compactly as $A\underset{\sim}{x} = \underset{\sim}{b}$, where $A[i,j] = a_{ij}$, $\underset{\sim}{x} = (x_1, \ldots, x_n)^T$ & $\underset{\sim}{b} = (b_1, \ldots, b_m)^T$.

We would like to know exactly when $A\underset{\sim}{x} = \underset{\sim}{b}$ has (a) no solutions (b) exactly one solution & (c) an infinite number of solutions. These questions will be answered in Ch.4.

Def. Let A be an $n \times n$ matrix. We say that A is *non-singular* if $A\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}$. Here $\underset{\sim}{x} = (x_1, \dots, x_n)^T$.
Also recall that A is *invertible* if there exists a matrix B such that $AB = I = BA$.

Theorem 9: Suppose A is an $n \times n$ matrix. Then A is invertible $\Leftrightarrow$ A is non-singular.

Proof: ($\Rightarrow$): Suppose A is invertible. Then we can find a matrix B such that $AB = I = BA$. Now if $A\underset{\sim}{x} = \underset{\sim}{0}$. The we have
$$\underset{\sim}{x} = I\underset{\sim}{x} = (BA)\underset{\sim}{x} = B(A\underset{\sim}{x}) = B(\underset{\sim}{0}) = \underset{\sim}{0}.$$
So $A\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} = 0$. Hence A is non-singular.

($\Leftarrow$) Suppose A is non-singular. Then the reduced row echelon form A_R of A must be I. (If A_R was not I, then some column of A_R must have no leading 1. And if we replace the zero in the diagonal position of that column by -1, we will get a nontrivial vector x_0 with $A x_0 = 0$). So we can find elementary matrices $E_k, E_{k-1}, \dots, E_1$ such that $BA = A_R = I$ and $B = E_k \cdots E_1$.
Hence
$$AB = I(AB) = (B^{-1}B)(AB) = B^{-1}(B(AB))$$
$$= B^{-1}(BA)B = B^{-1}IB = B^{-1}B = I.$$
Hence we have found a matrix B such that $BA = I = AB$. So A is invertible. □

Ex. If $A_R = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, you can take x_0 as $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ or as $\begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ ← diagonal position

diagonal position