Ch. 3 – The Trace & the Determinant

§1. Definition of the trace and determinant

Def Let A be an $n\times n$ matrix. We define the trace of A by $Tr(A) = \sum_{i=1}^{n} A[i,i]$.

Ex. 1 Let $A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 5 \\ 3 & 7 & 2 \end{bmatrix}$. Then

$Tr(A) = 2+1 = 3$ and $Tr(B) = 3+(-1)+2 = 4$.

Prop. 1 Let A and B be $n\times n$ matrices. Then

(a) $Tr(A^T) = Tr(A)$ (b) $Tr(\alpha A) = \alpha . Tr(A)$

(c) $Tr(A+B) = Tr(A) + Tr(B)$ (d) $Tr(AB) = Tr(BA)$

(e) $Tr(ABC) = Tr(CAB) = Tr(BCA)$

Proof:

(a) $Tr(A^T) = \sum_{i=1}^{n} (A^T)[i,i] = \sum_{i=1}^{n} A[i,i] = Tr(A)$

(b) $Tr(\alpha A) = \sum_{i=1}^{n} (\alpha A)[i,i] = \sum_{i=1}^{n} \alpha . A[i,i] = \alpha . \sum_{i=1}^{n} A[i,i] = Tr(A)$

(c) $Tr(A+B) = \sum_{i=1}^{n} (A+B)[i,i] = \sum_{i=1} A[i,i] + B[i,i]$

$= \sum_{i=1}^{n} A[i,i] + \sum_{i=1}^{n} B[i,i] = Tr(A) + Tr(B)$

(d) $Tr(AB) = \sum_{i=1}^{n} (AB)[i,i] = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} A[i,j] . B[j,i] \right\}$

$= \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} A[i,j] . B[j,i] \right\}$

$= \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} B[j,i] . A[i,j] \right\}$

$= \sum_{j=1}^{n} (BA)[j,j] = Tr(BA)$.

(e) $Tr(ABC) = Tr[(AB)C] = Tr[C(AB)] = Tr(CAB)$ by part (d)

$Tr(CAB) = Tr[(CA)B] = Tr[B(CA)] = Tr(BCA)$ by part (d) also.

Def. A <u>permutation</u> of the set $f: \{1,2,3,\ldots,n\}$ is any function $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ which is one-to-one and onto. So $\langle \sigma(1), \sigma(2), \ldots, \sigma(n) \rangle$ will be a rearrangement of the sequence $\langle 1,2,\ldots,n \rangle$ and we will this sequence to more easily specify σ. We will denote the set of all permutations on $\{1,2,3,\ldots,n\}$ by S_n.

Def. A <u>transposition</u> is any permutation formed by interchanging two entries of $\langle 1,2,\ldots,n \rangle$. Any permutation can be obtained by from $\langle 1,2,\ldots,n \rangle$ by repeatedly interchanging two entries. The smallest number of interchanges needed to transform $\langle 1,2,\ldots,n \rangle$ into σ is called the <u>transposition number</u> of σ and will be denoted by $t(\sigma)$

Ex. 2 Let $\sigma = \langle 3,1,2 \rangle$. Then we can get σ as follows

$$\langle 1,2,3 \rangle \longrightarrow \langle 3,2,1 \rangle \quad \text{switch } 1 \,\&\, 3$$
$$\longrightarrow \langle 3,1,2 \rangle \quad \text{switch } 1 \,\&\, 2.$$

So $t(\sigma) = 2$.

Def. Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the <u>determinant</u> of A by

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{t(\sigma)} \cdot \{ a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdot \cdots \cdot a_{n,\sigma(n)} \}$$

Ex. 3 Let $n=2$. Then $S_n = \{\langle 1,2 \rangle, \langle 2,1 \rangle\}$. Also $t(\langle 1,2 \rangle) = 0$ & $t(\langle 2,1 \rangle) = 1$. So

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{t(\langle 1,2 \rangle)} \{ a_{1,1} \cdot a_{2,2} \} + (-1)^{t\langle 2,1 \rangle} \cdot \{ a_{1,2} \cdot a_{2,1} \}$$
$$= a_{11} a_{22} - a_{12} a_{21}.$$

Ex. 4 Let $n=3$. Then $S_n = \{\langle 1,2,3\rangle, \langle 2,3,1\rangle, \langle 3,1,2\rangle, \langle 3,2,1\rangle, \langle 1,3,2\rangle, \langle 2,1,3\rangle\}$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} &(-1)^{t(\langle 1,2,3\rangle)} a_{11}a_{22}a_{33} + (-1)^{t(\langle 2,3,1\rangle)} a_{12}a_{23}a_{31} \\ &(-1)^{t(\langle 3,1,2\rangle)} a_{13}a_{21}a_{32} + (-1)^{t(\langle 3,2,1\rangle)} a_{13}a_{22}a_{31} \\ &(-1)^{t(\langle 1,3,2\rangle)} a_{11}a_{23}a_{32} + (-1)^{t(\langle 2,1,3\rangle)} a_{12}a_{21}a_{33} \end{aligned}$$

$$\begin{aligned} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

We can remember the determinant for 2×2 and 3×3 matrices as shown below.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

arrow to the right down $= +1$

arrow to the left down $= -1$

$$\left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \begin{aligned} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ &\quad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

For the 1×1 matrix, we simply have $\det([a_{11}]) = a_{11}$. Since $|S_n| = n!$, there will be 24 terms in the determinant for a 4×4 matrix $A = [a_{ij}]$.

Def. Let $A = [a_{ij}]$ be an $n\times n$ matrix. We define the (i,j) <u>minor submatrix</u> M_{ij} of A by

M_{ij} = matrix obtained by deleting row i and column j from A.

We define the (i,j) <u>cofactor</u> of the (i,j) entry a_{ij} in A by

$$A_{ij} = (-1)^{i+j}\det(M_{ij}).$$

Note that M_{ij} is an $(n-1)\times(n-1)$ matrix and $A_{ij} \in \mathbb{R}$,

Ex5 Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -5 \end{bmatrix}$. Then $M_{11} = \begin{bmatrix} -1 & 1 \\ 2 & -5 \end{bmatrix}$ & $M_{12} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}$.

So $A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 2 & -5 \end{vmatrix} = 3$ & $A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = -2$.

Def. The adjugate of an $n \times n$ matrix A is defined by

$$adg[A_{ij}] = [A_{ij}]^T$$

Ex.6 Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -5 \end{bmatrix}$. Then $adg(A) = \begin{bmatrix} 3 & 0 & 0 \\ 10 & -5 & -2 \\ 2 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 & 10 & 2 \\ 0 & -5 & -1 \\ 0 & -2 & -1 \end{bmatrix}$

and $A \cdot adg(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} 3 & 10 & 2 \\ 0 & -5 & -1 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$= 3 I_3$.

It is no coincidence that $det(A) = 3$, for we will see later that $adg(A) . A = \{det(A)\} I_n$.

Note: $\{A \cdot adg(A)\}[i,i] = \sum_{j=1}^{n} A[i,j] . adg[j,i]$

$= \sum_{j=1}^{n} a_{ij} \cdot A_{ij}$

$= a_{i1} A_{i1} + a_{i2} A_{i2} + a_{i3} A_{i3} + \cdots + a_{in} A_{in}$.

Theorem 2: For any $n \times n$ matrix A we have

$det(A) = \sum_{j=1}^{n} a_{ij} A_{ij}$ (Laplace's expansion along row i)

$det(A) = \sum_{i=1}^{n} a_{ij} A_{ij}$ (~~Laplace's expansion along column j~~)

Proof: The proof is by induction but is too long to give here.

§2. Properties of the determinant.

Def. Let $\mathbb{R}^*$ be the set of all finite sequences of real numbers. We define the <u>finite sum function</u> $\Sigma: \mathbb{R}^* \to \mathbb{R}$ by induction as follows.

(a) $\Sigma(\langle\ \rangle) = 0$ (b) $\Sigma(\langle a_1, \ldots, a_n, a_{n+1}\rangle) = \Sigma(\langle a_1, \ldots, a_n\rangle) + a_{n+1}$

Instead of writing $\Sigma(\langle a_1, \ldots, a_n\rangle)$ we usually write $\sum_{i=1}^{n} a_i$. We define the <u>finite product function</u> $\Pi: \mathbb{R}^* \to \mathbb{R}$ by induction also as follows.

(a) $\Pi(\langle\ \rangle) = 1$ (b) $\Pi(\langle a_1, \ldots, a_n, a_{n+1}\rangle) = \Pi(\langle a_1, \ldots, a_n\rangle) \cdot a_{n+1}$.

We usually write $\prod_{i=1}^{n} a_i$ instead of $\Pi(\langle a_1, \ldots, a_n\rangle)$.

Prop. 3. Let A be an $n \times n$ matrix.

(a) If A is upper triangular, then $\det(A) = \prod_{i=1}^{n} a_{ii}$.

(b) If A is lower triangular, then $\det(A) = \prod_{i=1}^{n} a_{ii}$.

(c) If A is a diagonal matrix, then $\det(A) = \prod_{i=1}^{n} a_{ii}$.

Proof (a) We will prove the result by induction on n.

<u>Basis</u>: If $n = 1$, then the result is true from the definition of the determinant. $\det[a_{11}] = a_{11} = \prod_{i=1}^{1} a_{ii}$.

<u>Ind. step</u>: Suppose that the result is true for all $(n-1)\times(n-1)$ matrices. Let A be any $n \times n$ matrix which is upper triangular. The by expanding $\det(A)$ about the first column we get

$$\begin{aligned}\det(A) &= a_{11} \cdot A_{11} + a_{21} A_{21} + a_{31} A_{31} + \cdots + a_{n1} A_{n1} \\ &= a_{11} \cdot (-1)^{1+1} \det(M_{11}) + 0 \cdot A_{21} + 0 \cdot A_{31} + \cdots + 0 \cdot A_{n1} \\ &= a_{11} \cdot \prod_{i=2}^{n} a_{ii} \quad \text{because } M_{11} \text{ is an } (n-1)\times(n-1) \text{ upper triangular matrix} \\ &= \prod_{i=1}^{n} a_{ii}\end{aligned}$$

So if the result is true for $(n-1)\times(n-1)$ matrices, it will be true for $n \times n$ matrices. Hence the result is true by The Princ. of Math Ind.

Proof: (b) & (c) Do for H.W.

Prop. 4 Let A be any $n \times n$ matrix. Then $\det(A^T) = \det(A)$

Proof: We will prove the result by induction on n.
Basis: If $n=1$, then $\det(A^T) = \det([a_{11}]^T) = \det([a_{11}]) = \det(A)$.
So the result is true for 1×1 matrices.
Ind. step: Suppose the result is true for $(n-1)\times(n-1)$ matrices.
Let $A = [a_{ij}]$ be any $n \times n$ matrices. Then by expanding along the first column of A^T we get

$$\det(A^T) = \sum_{j=1}^{n} (A^T)[j,1] \cdot (-1)^{j+1} \cdot \det[M_{j1}(A^T)]$$

$$= \sum_{j=1}^{n} A[1,j] \cdot (-1)^{1+j} \cdot \det([M_{1j}(A)]^T)$$

$$= \sum_{j=1}^{n} A[1,j] \cdot (-1)^{1+j} \cdot \det[M_{1j}(A)] \quad \text{by the Ind. hypothesis}$$

$$= \det(A) \quad \text{- expanded along row 1.}$$

So if the result is true for $n-1$, it will be true for n.
Hence the result is true for all $n \times n$ matrices by the Principle of Mathematical Induction.

Let us illustrate the key step of the proof when $n=3$,

$$\det(A^T) = \underbrace{\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}}_{A^T} = a_{11} \cdot (-1)^{1+1} \cdot \underbrace{\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix}}_{M_{11}(A^T)} + a_{12} \cdot (-1)^{2+1} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix}}_{M_{21}(A^T)} + a_{13} \cdot (-1)^{3+1} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{33} \end{vmatrix}}_{M_{31}(A^T)}$$

$$= a_{11}(-1)^{1+1} \cdot \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{M_{11}(A)} + a_{12} \cdot (-1)^{1+2} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{M_{12}(A)} + a_{13} \cdot (-1)^{1+3} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{33} \\ a_{31} & a_{33} \end{vmatrix}}_{M_{13}(A)}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det(A)$$

Prop. 5 : Let A' be the matrix formed by interchanging row i and row j of A. Then $\det(A') = -\det(A)$.

Proof: We will prove the result by induction on n.

Basis For $n=2$, we have only two rows. So $A' = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$, and so

$$\det(A') = (a_{21} \cdot a_{12} - a_{11}a_{22}) = -(a_{11}a_{22} - a_{12}a_{21}) = -\det(A)$$

Ind. step: Suppose the result is true for all $(n-1)\times(n-1)$ matrices. Let A be any $n \times n$ matrix with $n \geq 3$. Then expand $\det(A')$ along row k where $k \neq i$ or j.

$$\det(A') = a_{k1}A'_{k1} + a_{k2}A'_{k2} + \dots + a_{kn}A_{kn}$$
$$= a_{k1}\cdot(-1)^{k+1}\cdot\det[M_{k1}(A')] + \dots + a_{kn}\cdot(-1)^{k+n}\cdot\det[M_{kn}(A')]$$
$$= a_{k1}(-1)^{k+1}\cdot\{-\det[M_{k1}(A)]\} + \dots + a_{kn}\cdot(-1)^{k+n}\cdot\{-\det[M_{kn}(A)]\}$$
$$= -\{a_{k1}\cdot A_{k1} + a_{k2}A_{k2} + \dots + a_{kn}A_{kn}\}$$
$$= -\det(A).$$

So if the result is true for $n-1$, it will be true for n.

Concl. Hence by the Principle of Mathematical Induction, the result is true for all n.

Corollary 6 :

(a) Let A' be the matrix formed by interchanging two columns of A. Then $\det(A') = -\det(A)$.

(b) If A has two identical rows (or columns), then $\det(A) = 0$.

Proof: (a) $(A')^T$ will have two rows from A^T interchanged. So

$$\det(A') = \det[(A')^T] = -\det(A^T) \quad \text{by Proposition 4}$$
$$= -\det(A) \quad \text{by Proposition 3}.$$

Proof of Corollary 6

(b) Let A' be the matrix formed by interchanging those two identical rows or (identical columns). Then $A' = A$ because the rows (or columns) were identical.

So $\det(A) = \det(A')$

$= -\det(A)$ by Proposition 4

$\therefore\ 2\det(A) = 0$. Hence $\det(A) = 0$.

Prop. 7: Let A' be the matrix obtained by multiplying row i of A by α. Then $\det(A') = \alpha.\det(A)$.

Proof: If we expand $\det(A')$ along row i and take notice of the fact that all the minors $M_{i1}, \dots M_{in}$ of A' will be the same as those of A, we get

$$\begin{aligned} \det(A') &= (\alpha a_{i1})A'_{i1} + (\alpha a_{i2})A'_{i2} + \dots + (\alpha a_{in})A'_{in} \\ &= \alpha a_{i1} A_{i1} + \alpha a_{i2} A_{i2} + \dots + \alpha a_{in} A_{in} \\ &= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}) \\ &= \alpha.\det(A). \end{aligned}$$

Prop. 8: Let A' be the matrix obtained by replacing row i of A by row $i + \alpha$(row j). Then $\det(A') = \det(A)$

Proof: Let A'' be the matrix obtained by replacing row i of A by row j of A. Then $\det(A'') = 0$. Now if we expand $\det(A')$ along row i we get

$$\begin{aligned} \det(A') &= (a_{i1} + \alpha a_{j1})A'_{i1} + (a_{i2} + \alpha a_{j2})A'_{i2} + \dots + (a_{in} + \alpha a_{jn})A'_{in} \\ &= (a_{i1}A_{i1} + \dots + a_{in}A_{in}) + \alpha(a_{j1}A_{i1} + \dots + a_{jn}A_{in}) \\ &= \det(A) + \alpha(A''[i,1].A_{i1} + \dots + A''[i,n].A_{in}) \\ &= \det(A) + \alpha.\{A''[i,1].A''_{i1} + \dots + A''[i,n].A''_{in}\} \\ &= \det(A) + \alpha.\det(A'') = \det(A) + \alpha(0) = \det(A) \end{aligned}$$

§3 Determinants of products & related results

Theorem 9: Let E_1, E_2 and E_3 be elementary matrices of type I, II & III respectively. Then

(a) $\det(E_1) = -1$, $\det(E_2) = \alpha$, and $\det(E_3) = 1$

(b) $\det(E_p A) = \det(E_p) \cdot \det(A)$ for $p = 1, 2$ and 3

Proof: First observe that $\det(I_n) =$ prod. of diagonal elements $= 1$.

(a) $\det(E_1) = -\det(I_n) = -1$, because E_1 is obtained from I_n by interchanging two rows.

$\det(E_2) = \alpha \det(I_n) = \alpha$, because E_2 is obtained by multiplying a row by α.

Finally $\det(E_3) = \det(I_n) = 1$ because E_3 is obtained by replacing row i of I_n by row $i + \alpha$(row j) of I_n.

(b) This follows immediately from Prop. 5, 7 & 8 and part (a).

Recall that any $n \times n$ matrix A can be transformed into an upper triangular matrix A' by using Type I, type II & type III row operations. So

$$A' = E_1 E_2 \cdots E_k A$$

where each of the E_i's are elementary matrices. We can use this fact to find the determinant of A by using row operations.

Ex. 1
$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & -1 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & -2 & -4 \\ 0 & 4 & 5 \end{vmatrix} \begin{pmatrix} R2 := R2 - 3R1 \\ R3 := R3 + R1 \end{pmatrix} = (-2)\begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 5 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{vmatrix} (R3 := R3 - 4R2) = (-2)(-3)\begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 6.$$

Recall that an $n \times n$ matrix A was non-singular iff the equation $A\underset{\sim}{x} = \underset{\sim}{0}$ has no non-trivial solution. Recall also that A is invertible $\Leftrightarrow$ A is non-singular.

Theorem 10: The $n \times n$ matrix A is invertible $\Leftrightarrow \det(A) \neq 0$.

Proof: Let A_{RREF} be the reduced row echelon form of A. Then $A_{RREF} = E_k E_{k-1} \cdots E_1 A$ where the E_i's are elementary matrices. So

$$\begin{aligned} \det(A_{RREF}) &= \det(E_k E_{k-1} \cdots E_1 A) \\ &= \det(E_k) \cdot \det(E_{k-1} \cdots E_1 A) \\ &= \det(E_k) \cdot \det(E_{k-1}) \cdot \det(E_{k-2} \cdots E_1 A) \\ &= \cdots = \det(E_k) \cdot \det(E_{k-1}) \cdots \det(E_1) \cdot \det(A). \end{aligned}$$

Since $\det(E_i) \neq 0$ for each i, it follows that $\det(A_{RREF}) \neq 0$ iff $\det(A) \neq 0$.

Now if A is invertible, then A_{RREF} must be I_n and since $\det(A_{RREF}) = \det(I_n) = 1 \neq 0$, it follows that $\det(A) \neq 0$. And if A is not invertible, then A_{RREF} must have at least one row of zeros. So we get $\det(A_{RREF}) = 0$ and hence $\det(A) = 0$ also. Thus A is invertible $\Leftrightarrow \det(A) \neq 0$.

Theorem 11: Let A & B be $n \times n$ matrices. Then
(a) $\det(AB) = \det(A)\det(B)$, (b) $\det(BA) = \det(AB)$.

Proof (a) The proof splits into two cases:

Case (i): B is singular.

In this case we can find a non-trivial vector $\underset{\sim}{x_0}$ such that $B\underset{\sim}{x_0} = \underset{\sim}{0}$. But then $(AB)\underset{\sim}{x_0} = A(B\underset{\sim}{x_0})$ $= A(\underset{\sim}{0}) = \underset{\sim}{0}$. So AB is also singular. Hence

$$\begin{aligned} \det(AB) &= 0 \quad \text{(by Theorem 10)} \\ &= \det(A) . 0 \\ &= \det(A).\det(B) \quad \text{because } \det(B) = 0. \end{aligned}$$

Case (ii): B is non-singular.

In this case we can find elementary matrices $E_1, E_2, \dots, E_k$ such that $E_k E_{k-1} \dots E_1 B = I_n$. So $B = E_1^{-1} E_2^{-1} \dots E_k^{-1} I_n = F_1 F_2 \dots F_k$ where $F_i = E_i^{-1}$.

Hence

$$\begin{aligned} \det(AB) &= \det(AF_1 F_2 \cdots F_k) \\ &= \det[(AF_1F_2 \cdots F_k)^T] \\ &= \det(F_k^T F_{k-1}^T \cdots F_1^T A^T) \\ &= \det(F_k^T).\det(F_{k-1}^T) \cdots \det(F_1^T).\det(A^T) \\ &= \det(F_k).\det(F_{k-1}) \cdots \det(F_1).\det(A) \\ &= \det(F_1)\det(F_2) \dots \det(F_k).\det(A) \\ &= \det(F_1 F_2 \cdots F_k).\det(A) \\ &= \det(B).\det(A) = \det(A).\det(B). \end{aligned}$$

So in either case $\det(AB) = \det(A).\det(B)$.

(b) $\det(BA) = \det(B).\det(A) = \det(A).\det(B) = \det(AB)$.

Fact: If A & B are $n \times n$ matrices & $AB = I_n$, then $BA = I_n$.

Proof: Suppose $AB = I_n$. Then $\det(A).\det(B) = \det(AB) = 1$. So $\det(B) \neq 0$. Hence B is invertible. So we can find a matrix C such that $BC = I_n = CB$. Then

$BA = (BA)I_n = (BA)(BC) = B(A(BC)) = B((AB)C) = B(I_nC) = BC = I_n$.

__Prop. 12__: If A is an $n\times n$ matrix, then $A.\,adj(A) = det(A) I_n$.

__Proof:__ Recall that $adj(A)[i,j] = [A_{ij}]^T$. Now

$$\{A.\,adj(A)\}[i,j] = \sum_{k=1}^{n} A[i,k].\,(adj\,A)[k,j]$$
$$= \sum_{k=1}^{n} a_{ik} A_{jk}$$
$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{nj}$$
$$= \begin{cases} det(A) & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

So $A.\,adj(A) = det(A) I_n$ & so $A^{-1} = \frac{1}{det(A)}\, adj(A)$ if $det(A)\neq 0$.

__Theorem 13__ (Cramer's Rule)
Let A be an $n\times n$ non-singular matrix and $A(j/\underset{\sim}{b})$ be the matrix formed by replacing column j of A by $\underset{\sim}{b}$. Then the unique solution of the system $A\underset{\sim}{x} = \underset{\sim}{b}$ is given by $x_j = det[A(j/\underset{\sim}{b})] / det(A)$.

__Proof:__ We know that $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$. So $\underset{\sim}{x} = \frac{1}{det(A)}[adj(A)\,\underset{\sim}{b}]$

So $x_j = ([\text{row } j \text{ of } adj(A)]\,\underset{\sim}{b}) / det(A)$

$$= \left([A_{1j}\; A_{2j}\; \cdots\; A_{nj}]\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}\right) / det(A)$$
$$= (b_1 A_{1j} + b_2 A_{2j} + \dots + b_n A_{nj}) / det(A)$$
$$= \{\text{expansion of } det[A(j/b)] \text{ along column } j\} / det(A)$$
$$= det[A(j/\underset{\sim}{b})] / det(A)$$

__Ex.2__ Solve $\begin{bmatrix} 3 & 2 \\ 2 & 1\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = \begin{bmatrix} -1 \\ 2\end{bmatrix}$. $x_1 = \begin{vmatrix} -1 & 2 \\ 2 & 1\end{vmatrix} \Big/ \begin{vmatrix} 3 & 2 \\ 2 & 1\end{vmatrix} = \frac{-5}{-1} = 5$

$x_2 = \begin{vmatrix} 3 & -1 \\ 2 & 2\end{vmatrix} \Big/ \begin{vmatrix} 3 & 2 \\ 2 & 1\end{vmatrix} = \frac{8}{-1} = -8$. So $\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \begin{pmatrix} 5 \\ -8\end{pmatrix}$.

Def. Let A & B be $n \times n$ matrices. We say that A is <u>similar to</u> B if we can find an <u>invertible</u> matrix P such that $A = P^{-1}BP$.

<u>Prop. 14</u>: Let A, B & C be $n \times n$ matrices. Then
(a) A is similar to A (b) If A is similar B, the B is similar to A
(c) If A is similar to B & B is similar to C, then A is similar to C.

<u>Proof</u>: (a) $A = I^{-1}AI$, so A is similar to A.
(b) Suppose A is similar to B. Then we can find an invertible matrix P such that $A = P^{-1}BP$. Now P^{-1} is an invertible matrix and $(P^{-1})^{-1}A(P^{-1}) = PAP^{-1} = P(P^{-1}BP)P^{-1} = (P^{-1}P)B(PP^{-1}) = IBI = B$. So $B = (P^{-1})^{-1}A(P^{-1})$ and hence B is similar to A.
(c) Suppose A is similar to B & B is similar to C. Then we can find invertible matrices P & Q such that $A = P^{-1}BP$ & $B = Q^{-1}CQ$. Now QP is an invertible matrix and $(QP)^{-1}C(QP) = P^{-1}Q^{-1}CQP = P^{-1}(Q^{-1}CQ)P = P^{-1}BP = A$. Hence $A = (QP)^{-1}C(QP)$ & so A is similar to C.

<u>Prop 15</u>: Let A & B be $n \times n$ matrices. If A is similar to B, then
(a) $\det(A) = \det(B)$ (b) $Tr(A) = Tr(B)$.

<u>Proof</u>: Suppose A is similar to B. Then we can find an invertible matrix P such that $A = P^{-1}BP$. So
(a) $\det(A) = \det(P^{-1}(BP)) = \det((BP)P^{-1}) = \det(B(PP^{-1})) = \det(B)$,
& (b) $Tr(A) = Tr(P^{-1}BP) = Tr(P^{-1}(BP)) = Tr[(BP)P^{-1}] = Tr[B(PP^{-1})]$
$= Tr(B)$.

Ex.1 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix} \ \& \ BA = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$$

So $(AB)C = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix}$ & $C(BA) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

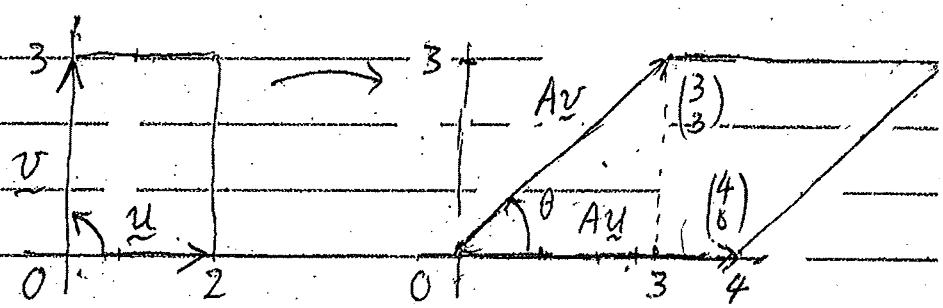
Hence $Tr(ABC) = 7 \neq 5 = Tr(CBA)$.

Geometrical interpretation of the determinant.

Fact: Let A be a 2×2 matrix and $\underset{\sim}{u}$ & $\underset{\sim}{v}$ be vectors in $\mathbb{R}^2$. Also let $P(\underset{\sim}{u}, \underset{\sim}{v})$ be the parallelogram determined by $\underset{\sim}{u}$ & $\underset{\sim}{v}$. Then $\det(A) = \dfrac{\text{signed area of } P(A\underset{\sim}{u}, A\underset{\sim}{v})}{\text{signed area of } P(\underset{\sim}{u}, \underset{\sim}{v})}$

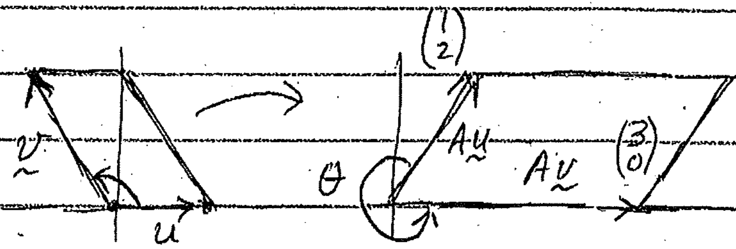
The sign of the area of $P(u,v)$ is the same as the sign of $\sin\theta$ where θ is the angle from $\underset{\sim}{u}$ to $\underset{\sim}{v}$, measured in the anti-clockwise direction

Ex.2 Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ & $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.



$$\det(A) = \frac{\text{signed area of } P(A\underset{\sim}{u}, A\underset{\sim}{v})}{\text{signed area of } P(\underset{\sim}{u}, \underset{\sim}{v})} = \frac{12}{6} = 2.$$

Ex.3 Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\underset{\sim}{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



Then $\det(B) = \dfrac{\text{signed area of } P(A\underset{\sim}{u}, A\underset{\sim}{v})}{\text{signed area of } P(\underset{\sim}{u}, \underset{\sim}{v})} = \dfrac{-6}{2} = -3$

Ex.4 $\det[\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}] = \underset{\sim}{u}\cdot(\underset{\sim}{v}\times\underset{\sim}{w}) = \underset{\sim}{v}\cdot(\underset{\sim}{w}\times\underset{\sim}{u}) = \underset{\sim}{w}\cdot(\underset{\sim}{u}\times\underset{\sim}{v})$ = signed vol. in 3-dim.