Ch.4 - Vector spaces & their bases

§1. Vector spaces & subspaces.

Def. Let V be a non-empty set. An n-ary operation on V is a function $f: V^n \to V$. Here $V^n = V \times V \times \cdots \times V$ (n times). We use the terms binary, unary and constant for 2-ary, 1-ary & 0-ary operations. When f is a binary operation, we usually write $\underset{\sim}{u} f \underset{\sim}{v}$ instead of $f(\underset{\sim}{u}, \underset{\sim}{v})$.

Def. A scalar operation on V over $\mathbb{R}$ is a function $g: \mathbb{R} \times V \to V$. Instead of $g(\alpha, \underset{\sim}{v})$ we usually write $\alpha \cdot \underset{\sim}{v}$ or just $\alpha \underset{\sim}{v}$.

Def. A vector space over $\mathbb{R}$ is an ordered 5-tuple $\mathcal{V} = (V, +, \cdot, -, \underset{\sim}{0})$ where V is a non-empty set, "$+$" is a binary operation on V, "$\cdot$" is a scalar operation on V over $\mathbb{R}$, "$-$" is a unary operation on V, and $\underset{\sim}{0}$ is a constant from V such that the following 8 axioms holds

A1 $(\forall \underset{\sim}{u})(\forall \underset{\sim}{v})(\forall \underset{\sim}{w})\,[(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})]$

A2 $(\forall \underset{\sim}{u})(\forall \underset{\sim}{v})\,[\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}]$

A3 $(\forall \underset{\sim}{u})\,[\underset{\sim}{0} + \underset{\sim}{u} = \underset{\sim}{u}]$

A4 $(\forall \underset{\sim}{u})\,[\underset{\sim}{u} + (-\underset{\sim}{u}) = \underset{\sim}{0}]$

A5 $(\forall \alpha \in \mathbb{R})(\forall \underset{\sim}{u})(\forall \underset{\sim}{v})\,[\alpha(\underset{\sim}{u} + \underset{\sim}{v}) = (\alpha \underset{\sim}{u}) + (\alpha \underset{\sim}{v})]$

A6 $(\forall \alpha \in \mathbb{R})(\forall \beta \in \mathbb{R})(\forall \underset{\sim}{u})\,[(\alpha + \beta)\underset{\sim}{u} = (\alpha \underset{\sim}{u}) + (\beta \underset{\sim}{u})]$

A7 $(\forall \alpha \in \mathbb{R})(\forall \beta \in \mathbb{R})(\forall \underset{\sim}{u})\,[(\alpha \beta)\underset{\sim}{u} = \alpha(\beta \underset{\sim}{u})]$

A8 $1(\underset{\sim}{u}) = \underset{\sim}{u}$.

Here "$\forall$" means "for all" & "$\exists$" means "there exists".

<u>Remark</u>: Since "+" and "·" are operations on V, we know that the following must always be true:

C1 If $\underset{\sim}{u}, \underset{\sim}{v} \in V$, then $\underset{\sim}{u} + \underset{\sim}{v} \in V$.

C2 If $\alpha \in \mathbb{R}$ & $\underset{\sim}{u} \in V$, then $\alpha \underset{\sim}{u} \in V$.

These are called closure properties of "+" & "·".
The elements of a vector space are called <u>vectors</u>.

<u>Examples of vectors spaces</u>

1. Let $\mathbb{R}^n$ be the set of all column vectors with n terms. Then $\langle \mathbb{R}^n, +, \cdot, -, \underset{\sim}{0} \rangle$ is a vector space over $\mathbb{R}$.
If $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, then $(-\underset{\sim}{u}) = \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix}$. Also $\underset{\sim}{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

2. Let $\mathbb{R}^{m \times n}$ be the set of all $m \times n$ matrices over $\mathbb{R}$. Then $\langle \mathbb{R}^{m \times n}, +, \cdot, -, \underset{\sim}{0} \rangle$ is a vector space over $\mathbb{R}$.
If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $(-A) = \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{bmatrix}$ & $\underset{\sim}{0} = O_{m,n}$

3. Let $\mathbb{R}[x] = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R} \text{ \& } n \in \mathbb{N}\}$. Then $\langle \mathbb{R}[x], +, \cdot, -, \underset{\sim}{0} \rangle$ is a vector space over $\mathbb{R}$. If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $-p(x) = -a_0 - \cdots - a_n x^n$ and $\underset{\sim}{0} = 0$. Also $(p+q)(x) = p(x) + q(x)$ and $\alpha . p(x) = (\alpha a_0) + (\alpha a_1) x + \cdots + (\alpha a_n) x^n$.

4. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Then $\langle \mathcal{F}(\mathbb{R}, \mathbb{R}), +, \cdot, -, \underset{\sim}{0} \rangle$ is a vector space over $\mathbb{R}$. If $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ then $(-f)$ is the function defined by $(-f)(x) = -[f(x)]$ and $\underset{\sim}{0}$ is the function defined by $\underset{\sim}{0}(x) = 0$ for each x. Also $(\alpha f)(x) = \alpha . f(x)$ and $(f+g)(x) = f(x) + g(x)$.

Prop. 1. Let $\mathcal{V} = \langle V, +, \cdot, -, \underset{\sim}{0} \rangle$ be a vector space and $\underset{\sim}{u}$ & $\underset{\sim}{v}$ be elements of V. Then

(a) $0(\underset{\sim}{u}) = \underset{\sim}{0}$

(b) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \underset{\sim}{v} = (-\underset{\sim}{u})$.

(c) $(-1)\underset{\sim}{u} = (-\underset{\sim}{u})$

Proof: First observe that

$$\underset{\sim}{u} = 1(\underset{\sim}{u}) = (1+0)\underset{\sim}{u} = 1(\underset{\sim}{u}) + 0(\underset{\sim}{u}) = \underset{\sim}{u} + 0(\underset{\sim}{u})$$

(a) So

$$\begin{aligned} 0(\underset{\sim}{u}) &= 0(\underset{\sim}{u}) + \underset{\sim}{0} \\ &= 0(\underset{\sim}{u}) + [\underset{\sim}{u} + (-\underset{\sim}{u})] \\ &= [0(\underset{\sim}{u}) + \underset{\sim}{u}] + (-\underset{\sim}{u}) \\ &= [\underset{\sim}{u} + 0(\underset{\sim}{u})] + (-\underset{\sim}{u}) \\ &= \underset{\sim}{u} + (-\underset{\sim}{u}) \quad \text{bec. } \underset{\sim}{u} = \underline{u} + 0(\underline{u}) \\ &= \underset{\sim}{0} \end{aligned}$$

Thus $0(\underset{\sim}{u}) = \underset{\sim}{0}$.

(b) Suppose $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0}$. Then

$$\begin{aligned} (-\underset{\sim}{u}) &= (-\underset{\sim}{u}) + \underset{\sim}{0} \\ &= (-\underset{\sim}{u}) + (\underset{\sim}{u} + \underset{\sim}{v}) \\ &= [(-\underset{\sim}{u}) + \underset{\sim}{u}] + \underset{\sim}{v} \\ &= [\underset{\sim}{u} + (-\underset{\sim}{u})] + \underset{\sim}{v} \\ &= \underset{\sim}{0} + \underset{\sim}{v} \\ &= \underset{\sim}{v}. \end{aligned}$$

So $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \underset{\sim}{v} = (-\underset{\sim}{u})$.

(c) Let $\underset{\sim}{v} = (-1)\underset{\sim}{u}$. Then

$$\begin{aligned} \underset{\sim}{u} + \underset{\sim}{v} &= \underset{\sim}{u} + (-1)\underset{\sim}{u} \\ &= (1)\underset{\sim}{u} + (-1)\underset{\sim}{u} \\ &= (1-1)\underset{\sim}{u} \\ &= 0(\underset{\sim}{u}) = \underset{\sim}{0} \end{aligned}$$

So $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0}$. Hence $\underset{\sim}{v} = (-\underset{\sim}{u})$. Thus $(-1)\underset{\sim}{u} = (-\underset{\sim}{u})$.

Def. Let S be a subset of V where V is the base set of a vector space $\mathcal{V} = \langle V, +, \cdot, -, \underset{\sim}{0} \rangle$. We say that $\mathcal{S} = \langle S, +, \cdot, -, \underset{\sim}{0} \rangle$ is a subspace of $\mathcal{V}$ if the following three results are satisfied

C0 S is non-empty.

C1 if $\underset{\sim}{u}, \underset{\sim}{v} \in S$, then $\underset{\sim}{u} + \underset{\sim}{v} \in S$.

C2 if $\alpha \in \mathbb{R}$ & $\underset{\sim}{u} \in S$, then $\alpha \underset{\sim}{u} \in S$.

If C0, C1 & C2 holds for the vectors in S, then $\mathcal{S} = \langle S, +, \cdot, -, \underset{\sim}{0} \rangle$ is a vector space in its own right with the inherited operations from $\mathcal{V}$. We usually say S is a subspace of V instead of going to all that trouble and saying that $\mathcal{S} = \langle S, +, \cdot, -, \underset{\sim}{0} \rangle$ is a subspace of $\mathcal{V} = \langle V, +, \cdot, -, \underset{\sim}{0} \rangle$.

Fact 2 If S is a subspace of V, then $\underset{\sim}{0} \in S$ and $\underset{\sim}{u} \in S \Rightarrow (-\underset{\sim}{u}) \in S$.

Proof: (a) Since S is non-empty, we can find at least one vector $\underset{\sim}{u} \in S$. Since $0 \in \mathbb{R}$ & $\underset{\sim}{u} \in S$, it follows that $\underset{\sim}{0} = 0(\underset{\sim}{u}) \in S$. $\therefore \underset{\sim}{0} \in S$

(b) Suppose $\underset{\sim}{u} \in S$. Then $-1 \in \mathbb{R}$ & $\underset{\sim}{u} \in S$. So $-\underset{\sim}{u} = (-1)\underset{\sim}{u} \in S$. $\therefore \underset{\sim}{u} \in S \Rightarrow (-\underset{\sim}{u}) \in S$.

Examples of subspaces

1. Let $S = \left\{ \begin{pmatrix} 2a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$. Now if $\underset{\sim}{u}, \underset{\sim}{v} \in S$, then we can find $a, b \in \mathbb{R}$ such that $\underset{\sim}{u} = \begin{pmatrix} 2a \\ a \end{pmatrix}$ & $\underset{\sim}{v} = \begin{pmatrix} 2b \\ b \end{pmatrix}$. So $\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} 2a \\ a \end{pmatrix} + \begin{pmatrix} 2b \\ b \end{pmatrix} = \begin{pmatrix} 2(a+b) \\ a+b \end{pmatrix} \in S$ & $\alpha \underset{\sim}{u} = \begin{pmatrix} 2(\alpha a) \\ (\alpha a) \end{pmatrix} \in S$. Since $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \in S$, $S \neq \emptyset$. ~~So S is a subspace of $\mathbb{R}^2$.~~

2. Let S be the set of all 2×2 matrices with trace equal 0. Then (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ because $Tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$. So $S \neq \emptyset$.

(b) Suppose $A, B \in S$. Then $Tr(A) = 0$ & $Tr(B) = 0$. So $Tr(A+B) = Tr(A) + Tr(B) = 0+0 = 0$. So $A+B \in S$.

(c) Suppose $\alpha \in \mathbb{R}$ and $A \in S$. Then $Tr(A) = 0$. So $Tr(\alpha A) = \alpha\, Tr(A) = \alpha \cdot 0 = 0$. So $\alpha A \in S$.

Hence S is a subspace of $\mathbb{R}^{2\times 2}$.

3. Let S be the set of all polynomials in $\mathbb{R}[x]$ in which the coefficients of x^{2n} are zero for each $n \in \mathbb{N}$.

(a) Since $x + x^3 \in S$, then $S \neq \emptyset$.

(b) Suppose $p(x), q(x) \in S$. Then the coefficients of x^{2n} in both $p(x)$ & $q(x)$ will be 0 for each $n \in \mathbb{N}$. So coefficients of x^{2n} in $(p+q)(x)$ will be 0 for each $n \in \mathbb{N}$. So $(p+q)(x) \in S$.

(c) Also if $\alpha \in \mathbb{R}$ & $p(x) \in S$, then the coefficients of x^{2n} in $p(x)$ will all be zeros for each $n \in \mathbb{N}$. So the coefficients of $(\alpha p)(x)$ will be 0 for each $n \in \mathbb{N}$. So $(\alpha p)(x) \in S$. Hence S is a subspace of $\mathbb{R}[x]$.

4. Let S be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) Since the function $f(x) \equiv 0$ for each $x \in \mathbb{R}$ is continuous, $S \neq \emptyset$

(b) If $f, g \in S$, then $f+g \in S$, because the sum of two continuous.

(c) If $\alpha \in \mathbb{R}$ & $f \in S$, then $\alpha f \in S$, because αf will be continuous if f is continuous. So S is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

§2. Linear combinations & linear independence.

Def. Let $v_1, v_2, \ldots, v_k$ be k vectors in a vector space V. A linear combination of $v_1, \ldots, v_k$ is any expression of the form $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k$ where $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.

The span (or linear span) of $(v_1, \ldots, v_k)$ is defined by $\text{span}(v_1, \ldots, v_k)$ = the set of all linear combinations of the vectors $v_1, \ldots, v_k$

Ex. 1 Let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ & $v_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ be vectors in $\mathbb{R}^2$. Then

$$5 v_1 + (-4) v_2 = 5\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-4)\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} + \begin{pmatrix} 8 \\ -12 \end{pmatrix} = \begin{pmatrix} 13 \\ -17 \end{pmatrix}$$

is a linear combination of v_1 & v_2. Also

$$\text{span}(v_1, v_2) = \left\{ \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta\begin{pmatrix} -2 \\ 3 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

Prop. 3: If $v_1, \ldots, v_k$ are vectors in a vector space V, then $\text{span}(v_1, \ldots, v_k)$ is a subspace of V.

Proof: Let $S = \text{span}(v_1, \ldots, v_k)$. Then

(a) $0 \in S$ because $0 = 0.v_1 + 0.v_2 + \ldots + 0.v_k$ is a linear combination

(b) Suppose $u, w \in S$. Then we can find scalars $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ such that

$$u = \alpha_1 v_1 + \ldots + \alpha_k v_k \quad \& \quad w = \beta_1 v_1 + \cdots + \beta_k v_k.$$

So $u + w = (\alpha_1 + \beta_1) v_1 + \ldots + (\alpha_k + \beta_k) v_k \in S$.

(c) Also if $\alpha \in \mathbb{R}$ & $u \in S$, then $\alpha u = (\alpha a_1) v_1 + \ldots + (\alpha a_k) v_k \in S$. Hence S is a subspace of V.

Qu: How can we tell if a vector $\underset{\sim}{u}$ is in span$(\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k)$?

Ans: Try to find $c_1, \dots, c_k \in \mathbb{R}$ such that $\underset{\sim}{u} = c_1 \underset{\sim}{v}_1 + \dots + c_k \underset{\sim}{v}_k$. If we succeed, then $\underset{\sim}{u} \in$ span$(\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k)$. If we show that this is not possible, then $\underset{\sim}{u} \notin$ span$(\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k)$.

Ex. 2 Let $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underset{\sim}{v}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{u} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Is $\underset{\sim}{u} \in$ span$(\underset{\sim}{v}_1, \underset{\sim}{v}_2)$

Sol. Suppose $\underset{\sim}{u} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2$. Then

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

So
$$\left.\begin{aligned} c_1 - 2c_2 &= 5 \\ -c_1 + 3c_2 &= 4 \end{aligned}\right\} \quad \begin{aligned} &E1 \\ &E2 \end{aligned}$$

$$\therefore \quad \begin{aligned} c_1 - 2c_2 &= 5 \\ c_2 &= 9 \qquad E2 := E2 + E1 \end{aligned}$$

$\therefore \quad c_1 = 5 + 2c_2 = 5 + 18 = 23$. So $\underset{\sim}{u} = 23 \underset{\sim}{v}_1 + 9 \underset{\sim}{v}_2$

Check: $23 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 9 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 23 - 18 \\ -23 + 27 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Ex. 3 Let $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, $\underset{\sim}{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\underset{\sim}{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. Is $\underset{\sim}{u} \in$ span(v_1, v_2)?

Suppose $\underset{\sim}{u} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2$. Then $c_1 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$$\left.\begin{aligned} c_1 - 2c_2 &= 1 \\ -2c_1 + c_2 &= -2 \\ -c_1 + c_2 &= 3 \end{aligned}\right\} \rightarrow \left.\begin{aligned} c_1 - 2c_2 &= 1 \\ -3c_2 &= 0 \\ -c_2 &= 4 \end{aligned}\right\} \quad \begin{aligned} &E2 := E2 + 2E1 \\ &E3 := E3 + E1 \end{aligned}$$

$$\rightarrow \left.\begin{aligned} c_1 - 2c_2 &= 1 \\ 0 &= -12 \\ -c_2 &= 4 \end{aligned}\right\} \quad E2 := E2 - 3E3$$

But $0=-12$ is not possible. So $\underset{\sim}{u} \notin span(\underset{\sim}{v}_1, \underset{\sim}{v}_2)$.

Def. Let $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_k$ be k vectors in a vector space V. We say that $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k$ is <u>linearly independent</u> if $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + \cdots + c_k\underset{\sim}{v}_k = \underset{\sim}{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0$.

We say that $\underset{\sim}{v}_1, \ldots, v_k$ is <u>linearly dependent</u> if $\underset{\sim}{v}_1, \ldots, v_k$ is <u>not</u> linearly independent. Observe that this means that we can find scalars $c_1, \ldots, c_k$ with at least one $c_i \neq 0$, such that $c_1 v_1 + c_2 v_2 + \cdots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}$.

Ex. 4 Let $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ & $\underset{\sim}{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Are v_1, v_2 linearly independent?

Sol. Suppose $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 = \underset{\sim}{0}$. The $c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
So

$$\left.\begin{aligned} c_1 - 2c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} c_1 - 2c_2 &= 0 \\ -c_2 &= 0 \end{aligned}\right\} E2 := E2 + E1$$

$$\rightarrow \left.\begin{aligned} c_1 \quad &= 0 \\ -c_2 &= 0 \end{aligned}\right\} E1 := E1 - 2E$$

$\therefore$ $c_1 = c_2 = 0$. Hence $\underset{\sim}{v}_1$ & $\underset{\sim}{v}_2$ are linearly indep.

Ex. 5 Let $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underset{\sim}{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ & $\underset{\sim}{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Is $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ a linearly independent set?

Sol. Suppose $c_1 v_1 + c_2 v_2 + c_3 \underset{\sim}{v}_3 = \underset{\sim}{0}$. Then

Ex. 5

$$\left.\begin{array}{l} c_1 - c_2 + c_3 = 0 \\ 0\cdot c_1 + 2c_2 + 2c_3 = 0 \\ c_1 + c_2 + 3c_3 = 0 \end{array}\right\} \rightarrow \left.\begin{array}{r} c_1 - c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \\ 2c_2 + 2c_3 = 0 \end{array}\right\} \begin{array}{l} \\ E2 := (1/2)E2 \\ E3 := E3 - E1 \end{array}$$

$$\rightarrow \left.\begin{array}{r} c_1 - c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \\ 0 = 0 \end{array}\right\} \begin{array}{l} \\ \\ E3 := E3 - E2 \end{array}$$

$$\rightarrow \left.\begin{array}{r} c_1 \qquad + 2c_3 = 0 \\ c_2 + c_2 = 0 \end{array}\right\} \begin{array}{l} E1 := E1 + E2 \\ \\ \end{array}$$

$\therefore$ $c_1 = -2c_3$ & $c_2 = -c_3$. So if we take $c_3 = 1$, we will get $c_1 = -2$ & $c_2 = -1$. Thus

$$(-2)\begin{pmatrix}1\\0\\1\end{pmatrix} + (-1)\begin{pmatrix}-1\\2\\1\end{pmatrix} + 1\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So $(-2)\underline{v}_1 + (-1)\underline{v}_2 + (1)\underline{v}_3 = \underline{0}$. Hence $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is not a linearly independent set of vectors.

Theorem 4: Let $S = \text{span}\{\underline{v}_1, \ldots, \underline{v}_k\}$ & $\underline{u}_0 \in S$. Then $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is linearly independent $\Longleftrightarrow$ $\underline{u}_0$ can be expressed in only one way as a linear combination of $\{\underline{v}_1, \ldots, \underline{v}_k\}$

Proof. ($\Rightarrow$) Suppose $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is linearly independent. Then $c_1\underline{v}_1 + \cdots + c_k\underline{v}_k = \underline{0} \Rightarrow c_1 = c_2 = \cdots = c_k = 0$. Now if $\underline{u}_0 = \alpha_1\underline{v}_1 + \cdots + \alpha_k\underline{v}_k$ and $\underline{u}_0 = \beta_1\underline{v}_1 + \cdots + \beta_k\underline{v}_k$, then

$$(\alpha_1 - \beta_1)\underline{v}_1 + \cdots + (\alpha_k - \beta_k)\underline{v}_k = (\alpha_1\underline{v}_1 + \cdots + \alpha_k\underline{v}_k) - (\beta_1\underline{v}_1 + \cdots + \beta_k\underline{v}_k)$$
$$= \underline{u}_0 - \underline{u}_0 = \underline{0}.$$

So $\alpha_i - \beta_i = 0$ for each i. Hence $\underline{u}_0$ can be expressed in only one way as a linear combination of $\{\underline{v}_1, \ldots, \underline{v}_k\}$

($\Leftarrow$) Suppose $\underline{u}_0$ can be expressed in only way as a lin. comb. of $\{\underline{v}_1, \ldots, \underline{v}_k\}$, let's say $\underline{u}_0 = \alpha_1\underline{v}_1 + \cdots + \alpha_k\underline{v}_k$. Now if $c_1\underline{v}_1 + \cdots + c_k\underline{v}_k = \underline{0}$ then $\underline{u}_0 = \underline{u}_0 + \underline{0} = (\alpha_1 + c_1)\underline{v}_1 + \cdots + (\alpha_k + c_k)\underline{v}_k$.

Since $\underset{\sim}{v}_0$ can be expressed in only one way as a linear combination of $v_1, \dots, v_k$ we must have $c_1 = c_2 = \cdots = c_k = 0$. Hence $\{\underline{v}_1, \dots, v_k\}$ is linearly independent.

So far we have only defined what is span $(\underline{v}_1, \dots, \underline{v}_k)$ for a finite set $\{v_1, \dots, \underset{\sim}{v}_k\}$ of vectors from V. We have also only defined what it means for a finite set $\{v_1, \dots, v_k\}$ of vectors from V, to be linearly independent.

Recall that

$$\text{span}\{\underline{v}_1, \dots, \underline{v}_k\} = \{\alpha_1 \underline{v}_1 + \cdots + \alpha_k \underline{v}_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$$

and that $\{\underline{v}_1, \dots, v_k\}$ is linearly independent if

$$c_1 \underset{\sim}{v}_1 + c_2 v_2 + \dots + c_k \underline{v}_k = \underset{\sim}{0} \Rightarrow c_1 = c_2 = \dots = c_k = 0.$$

Def. Let S be any set (finite or infinite) of vectors from V. We define span(S) to be the set of all finite linear combinations of vectors from S. So

$$\text{span}(S) = \{\alpha_1 \underset{\sim}{v}_1 + \cdots + \alpha_k \underset{\sim}{v}_k : \{\underset{\sim}{v}_1, \dots, v_k\} \text{ is a finite subset of } S \text{ and } \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$$

Def. Let S be any set (finite or infinite) of vectors from V. We say that S is linearly independent if for each finite subset $\{\underline{v}_1, \dots, \underline{v}_k\}$ of S,

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_k v_k = \underset{\sim}{0} \Rightarrow c_1 = c_2 = \dots = c_k = 0.$$

Ex. 6 span$(\{x^k : k \in \mathbb{N}\})$ = set of all polyonomials in $\mathbb{R}[x]$

$S = \{x^k : k \in \mathbb{N}\}$ is a linearly independent set of vectors in $\mathbb{R}[x]$.

§3. Bases & dimension of a vector space:

Def. Let B be a set of vectors from a vector space V. We say that B is a <u>basis</u> of V if
(a) $\text{span}(B) = V$, and (b) B is linearly independent.

Ex 1(a) Let $E_2 = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$. Then E_2 is a basis of $\mathbb{R}^2$.

If $B_2 = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \right\}$, then B_2 is another basis of $\mathbb{R}^2$.

Ex 1(b) Let $E_3 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ & $B_3 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$. Then E_3 & B_3 are both bases of $\mathbb{R}^3$.

Ex 1(c) Let $B = \left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$. Then B is a basis of $\mathbb{R}^{2\times 2}$.

Ex 1(d) Let $e_1 = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}$, $\ldots$, $e_n = \begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}$. Then $E_n = \{e_1, \ldots, e_n\}$ is the <u>standard basis</u> of $\mathbb{R}^n$

Ex 1(e) Let $B_{n+1} = \{x^0, x^1, \ldots, x^n\}$. Then B_{n+1} is the <u>standard basis</u> of $\mathbb{R}_n[x]$ = the set of all polynomials in x (with real coefficients) of degree $\leq n$.

Ex 1(f) Let $B = \{x^k : k \in \mathbb{N}\} = \{x^0, x^1, x^2, \ldots, x^k, \ldots\}$. Then B is the standard basis of $\mathbb{R}[x]$.
$B' = \{(1+x)^k : k \in \mathbb{N}\}$ is another basis of $\mathbb{R}[x]$.

Theorem 5 Suppose $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_m\}$ is a basis of V and $m < n$. Then any set of n vectors from V is linearly dependent.

Proof. Let $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_n$ be n vectors from V. Since $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_m\}$ is a basis of V, we can express each $\underset{\sim}{u}_j$ uniquely as

$$\underset{\sim}{u}_j = a_{1j}\underset{\sim}{v}_1 + a_{2j}\underset{\sim}{v}_2 + \cdots + a_{mj}\underset{\sim}{v}_m, \quad (j = 1, \ldots, n).$$

So the equation

$$c_1\underset{\sim}{u}_1 + c_2\underset{\sim}{u}_2 + \cdots + c_n\underset{\sim}{u}_n = \underset{\sim}{0} \qquad (*)$$

becomes

$$\begin{aligned} & c_1(a_{11}\underset{\sim}{v}_1 + a_{21}\underset{\sim}{v}_2 + \cdots + a_{m1}\underset{\sim}{v}_m) \\ +\ & c_2(a_{12}\underset{\sim}{v}_1 + a_{22}\underset{\sim}{v}_2 + \cdots + a_{m2}\underset{\sim}{v}_m) \\ & \vdots \\ +\ & c_n(a_{1n}\underset{\sim}{v}_1 + a_{2n}\underset{\sim}{v}_2 + \cdots + a_{mn}\underset{\sim}{v}_m) = \underset{\sim}{0} \end{aligned}$$

$$\begin{aligned} \therefore\ & (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n)\underset{\sim}{v}_1 \\ +\ & (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\underset{\sim}{v}_2 \\ & \vdots \\ +\ & (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\underset{\sim}{v}_m = \underset{\sim}{0} \end{aligned}$$

Since $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_m\}$ is linearly independent it follows that

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n &= 0 \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n &= 0 \end{aligned}$$

But this is a homogeneous system of m equations with n unknowns. So it has a non-trivial solution. So $(*)$ has a non-trivial solution & hence $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\}$ is lin. dep.

Corollary 6 If $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\}$ and $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_m\}$ are both bases of a vector space V, then $m = n$.

Proof: Since $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_m\}$ is a basis of V and $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\}$ is linearly independent, we must have $m \not< n$ by Theorem 5. So $n \leq m$. Also since $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n\}$ is a basis of V and $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_m\}$ is linearly independent, we must also have $n \not< m$ by Theorem 5. So $m \leq n$. $\therefore m = n$.

Def. Let V be a vector space. We say that V is finite-dimensional if it has a finite basis. If V has no finite basis, then V is said to be infinite-dimensional.

Def. The dimension of a finite-dimensional vector space V is defined to be the number of elements in any basis of V and denoted by $\dim(V)$

Ex. 2 (a) $\mathbb{R}^2$ is finite dimensional & $\dim(\mathbb{R}^2) = 2$

(b) $\mathbb{R}^3$ is " & $\dim(\mathbb{R}^3) = 3$

(c) $\mathbb{R}^{2\times2}$ is " & $\dim(\mathbb{R}^{2\times2}) = 4$

(d) $\mathbb{R}^n$ is " & $\dim(\mathbb{R}^n) = n$

(e) $\mathbb{R}^{m\times n}$ is " & $\dim(\mathbb{R}^{m\times n}) = m \cdot n$

(f) $\mathbb{R}_n[x]$ is " & $\dim(\mathbb{R}_n[x]) = n+1$.

Ex. 3 (a) $\mathbb{R}[x]$ is infinite-dimensional.

(b) $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is infinite-dimensional.

(c) $\{\underset{\sim}{0}\}$ is finite-dimensional — $\emptyset$ is a basis of $\{\underset{\sim}{0}\}$

§4. The Four Fundamental Subspaces of a matrix.

Let A be an $m \times n$ matrix. Recall that we can write A as a row of n column vectors from $\mathbb{R}^m_\downarrow$ or as a column of m row vectors from $\mathbb{R}^n_\rightarrow$.

$$A = \begin{bmatrix} \underset{\sim}{c}_1(A), \cdots, \underset{\sim}{c}_n(A) \end{bmatrix} \qquad A = \begin{bmatrix} \vec{r}_1(A) \\ \vdots \\ \vec{r}_m(A) \end{bmatrix}$$

Ex. 1 Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & -1 & 4 \end{bmatrix}$. Then

$$A = \begin{bmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} & \begin{pmatrix} 2 \\ -1 \end{pmatrix} & \begin{pmatrix} -2 \\ 4 \end{pmatrix} \end{bmatrix} \ \& \ A = \begin{bmatrix} (1, 2, -2) \\ (3, -1, 4) \end{bmatrix}$$

Def. We define the column space & row space of A by

$$ColSp(A) = span\{\underset{\sim}{c}_1(A), \ldots, \underset{\sim}{c}_n(A)\}$$
$$RowSp(A) = span\{\vec{r}_1(A), \ldots, \vec{r}_m(A)\}.$$

Note that $ColSp(A)$ is a subspace of $\mathbb{R}^m_\downarrow$ and $RowSp(A)$ is a subspace of $\mathbb{R}^n_\rightarrow$.

Ex. 2 Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & -1 & 4 \end{bmatrix}$. Then

(a) $$ColSp(A) = \left\{\alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} -2 \\ 4 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} 1 & 2 & -2 \\ 3 & -1 & 4 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}\right\}$$

$$= \{A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^3_\downarrow\}$$

(b) $$RowSp(A) = \{\alpha(1, 2, -2) + \beta(3, -1, 4) : \alpha, \beta \in \mathbb{R}\}$$

$$= \left\{ (\alpha, \beta) \begin{bmatrix} 1 & 2 & -2 \\ 3 & -1 & 4 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$$= \{ \vec{x} A : \vec{x} \in \mathbb{R}^3_{\rightarrow} \}$$

<u>Prop 7</u>: If A is any $m \times n$ matrix, then

(a) $ColSp(A) = \{ A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n_{\downarrow} \}$ and

(b) $RowSp(A) = \{ \vec{x} A : \vec{x} \in \mathbb{R}^m_{\rightarrow} \}$.

<u>Proof.</u> (a) $\{ A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n_{\downarrow} \} = \{ x_1 \underset{\sim}{c}_1(A) + x_2 \underset{\sim}{c}_2(A) + \cdots + x_n \underset{\sim}{c}_n(A) : x_i \in \mathbb{R} \}$

$= span\{ \underset{\sim}{c}_1(A), \ldots, \underset{\sim}{c}_n(A) \}$

(b) $\{ \vec{x} A : \vec{x} \in \mathbb{R}^m_{\rightarrow} \} = \{ x_1 \vec{r}_1(A) + x_2 \vec{r}_2(A) : x_i \in \mathbb{R} \}$

$= span\{ \vec{r}_1(A), \vec{r}_2(A) \}$.

<u>Def.</u> Let A be an $m \times n$ matrix. We defined the <u>Null space</u> & the **<u>co-null</u>** space of A by

$$Null(A) = \{ \underset{\sim}{x} \in \mathbb{R}^n_{\downarrow} : A\underset{\sim}{x} = \underset{\sim}{0} \}$$

$$Co\text{-}null(A) = \{ \vec{x} \in \mathbb{R}^m_{\rightarrow} : \vec{x} A = \vec{0} \}.$$

<u>Prop. 8</u>: Let A be any $m \times n$ matrix. Then

(a) $Null(A)$ is a subspace of $\mathbb{R}^n_{\downarrow}$

(b) $CoNull(A)$ is a subspace of $\mathbb{R}^m_{\rightarrow}$.

<u>Proof.</u> (a) $A \underset{\sim}{0}_n = \underset{\sim}{0}_m$. So $\underset{\sim}{0}_n \in Null(A)$. $\therefore$ $Null(A) \neq \emptyset$

Now suppose $\underset{\sim}{x}, \underset{\sim}{y} \in Null(A)$. Then $A\underset{\sim}{x} = \underset{\sim}{0}$ & $A\underset{\sim}{y} = \underset{\sim}{0}$.

So $A(\underset{\sim}{x} + \underset{\sim}{y}) = A\underset{\sim}{x} + A\underset{\sim}{y} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$. $\therefore$ $\underset{\sim}{x} + \underset{\sim}{y} \in Null(A)$

Finally suppose $\alpha \in \mathbb{R}$ and $\underset{\sim}{x} \in Null(A)$. Then

$A(\alpha \underset{\sim}{x}) = \alpha(A\underset{\sim}{x}) = \alpha(\underset{\sim}{0}) = \underset{\sim}{0}$ $\therefore$ $\alpha \underset{\sim}{x} \in Null(A)$.

Hence $Null(A)$ is a subspace of $\mathbb{R}^n_{\downarrow}$.

(b) Do for Homework.

Ex 3. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$. Find Null(A) & CoNull(A).

Sol (a) Suppose $A\underset{\sim}{x} = \underset{\sim}{0}$. Then $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R2 := R2 + R1 \\ R3 := R3 - R1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R1 := R1 + 2R2 \\ \\ R3 := R3 - 2R2 \end{array}$$

$\therefore x_3 = \alpha,\ x_2 = -x_3 = -\alpha;\ \therefore x_1 = -3x_3 = -3\alpha$

$\therefore \text{Null}(A) = \left\{ \begin{pmatrix} -3\alpha \\ -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

(b) Suppose $\vec{x}A = \vec{0}$. Then $(\vec{x}A)^T = (\vec{0})^T$. So $A^T(\vec{x})^T = \underset{\sim}{0}$

Let $\underset{\sim}{y} = (\vec{x})^T$. Then $\begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -2 & 3 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R2 := R2 + 2R1 \\ R3 := R3 - R1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R1 := R1 + R2 \\ \\ R3 := R3 - R2 \end{array}$$

$\therefore y_3 = \beta,\ y_2 = -y_3 = -\beta,\ y_1 = -2y_3 = -2\beta$. So

$\text{CoNull}(A) = \{(-2\beta, -\beta, \beta) : \beta \in \mathbb{R}\} = \{\beta(-2, -1, 1) : \beta \in \mathbb{R}\}$

Prop. 9: Let A be any $m \times n$ matrix. Then

(a) $[RowSp(A)]^T = ColSp(A^T)$ (b) $[CoNull(A)]^T = Null(A^T)$

Proof:

(a) $[RowSp(A)]^T = span\{\vec{r}_1(A), \ldots, \vec{r}_m(A)\}^T = span\{\vec{r}_1(A)^T, \ldots, \vec{r}_m(A)^T\}$
$= span\{\underset{\sim}{c}_1(A^T), \ldots, \underset{\sim}{c}_m(A^T)\} = ColSp(A^T)$

(b) $[CoNull(A)]^T = \{\vec{x} \in \mathbb{R}^m_{\rightarrow} : \vec{x}A = \vec{0}\}^T = \{(\vec{x})^T \in \mathbb{R}^m_{\downarrow} : \vec{x}A = \vec{0}\}$
$= \{(\vec{x})^T \in \mathbb{R}^m_{\downarrow} : (A\vec{x})^T = \underset{\sim}{0}\} = \{(\vec{x})^T \in \mathbb{R}^m_{\downarrow} : A^T(\vec{x})^T = \underset{\sim}{0}\}$
$= \{\underset{\sim}{Y} \in \mathbb{R}^m_{\downarrow} : (A^T)\underset{\sim}{Y} = \underset{\sim}{0}\} = Null(A^T)$.

Def. The matrix A is said to be row equivalent to B if we can obtain A by a finite number of type I, II or III row operations on B.

Note: If A is row equivalent to B, then we can find elementary matrices $E_1, \ldots, E_k$ such that $A = E_k \cdots E_2 E_1 B$. So $B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A$. Hence B is row equivent to A.

Prop. 10: If A is row equivalent to B and A is $m \times n$, then $RowSp(A) = RowSp(B)$.

Proof: Suppose A is row equivalent to B. Then $A = PB$ where P is a finite product of elementary matrices. So $\vec{r}_i(A) = p_{i1}\vec{r}_1(B) + p_{i2}\vec{r}_2(B) + \cdots + p_{im}\vec{r}_m(B)$. Hence $RowSp(A) \subseteq RowSp(B)$. Since B is also row equivalent to A, $RowSp(B) \subseteq RowSp(A)$. Hence $RowSp(A) = RowSp(B)$.

§5. Row rank, column rank, nullity & co-nullity

Def. We define the row rank & column rank of A by

$$\text{row rank}(A) = \dim[\text{RowSp}(A)]$$
$$\text{col rank}(A) = \dim[\text{Col Sp}(A)]$$

We also define the nullity & co-nullity of A

$$\text{nullity}(A) = \dim[\text{Null}(A)]$$
$$\text{co-nullity}(A) = \dim[\text{Co-Null}(A)]$$

Theorem 11 (Row rank = Col rank Theorem)
Let A be any $m \times n$ matrix. Then
$$\text{row rank}(A) = \text{col rank}(A).$$

Proof. Suppose $A \neq O_{m,n}$. Let $\{\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_r\}$ be a basis of RowSp(A). Put $\underset{\sim}{Y}_i = A(\vec{X}_i)^T$. We claim that $\{\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_r\}$ is linearly independent. Indeed, suppose $c_1 \underset{\sim}{Y}_1 + c_2 \underset{\sim}{Y}_2 + \cdots + c_r \underset{\sim}{Y}_r = \underset{\sim}{0}$. Then

$$\begin{aligned}
\underset{\sim}{0} &= c_1\{A(\vec{X}_1)^T\} + c_2\{A(\vec{X}_2)^T\} + \cdots + c_r\{A(\vec{X}_r)^T\} \\
&= A\{c_1(\vec{X}_1)^T\} + A\{c_2(\vec{X}_2^T)\} + \cdots + A\{c_r(\vec{X}_r)^T\} \\
&= A\{c_1(\vec{X}_1)^T + c_2(\vec{X}_2)^T + \cdots + c_r(\vec{X}_r)^T\} \\
&= A\underset{\sim}{v}, \quad \text{where } \underset{\sim}{v} = c_1(\vec{X}_1)^T + \cdots + c_r(\vec{X}_r)^T.
\end{aligned}$$

So $A\underset{\sim}{v} = \underset{\sim}{0}$. Hence $\vec{r}_i(A)\underset{\sim}{v} = 0$ for each row $\vec{r}_i(A)$ of A. Consequently $\vec{u}\,\underset{\sim}{v} = 0$ for any linear combinations $\vec{u}$ of the rows of A. But $\underset{\sim}{v}^T$ is a linear combination of the rows of A. So $\underset{\sim}{v}^T\underset{\sim}{v} = 0$. Hence $\underset{\sim}{v} = \vec{0}$. So $c_1(\vec{X}_1)^T + \cdots + c_r(\vec{X}_r)^T = \underset{\sim}{0}$
$\therefore$ $c_1(\vec{X}_1) + \cdots + c_r(\vec{X}_r) = \vec{0}$. Since $\{\vec{X}_1, \ldots, \vec{X}_r\}$ was a basis, $\{\vec{X}_1, \ldots, \vec{X}_n\}$ is linearly independent. Hence $c_1 = c_2 = \cdots = c_r = 0$. $\therefore$ $\{\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_r\}$ is lin. indep.

But $Y_i = A(\vec{x}_i)^T \in ColSp(A)$ for each i. Hence $\dim[ColSp(A)] \geq r$. Thus col rank$(A) \geq$ row rank(A).

Now row rank(A) = col rank(A^T) & col rank(A) = row rank(A^T) because $[RowSp(A)]^T = ColSp(A^T)$ & $[ColSp(A)]^T = RowSp(A^T)$ — see Proposition 9. Hence

$$\begin{aligned} \text{row rank}(A) &= \text{col rank}(A^T) \\ &\geq \text{row rank}(A^T) \quad \text{from above} \\ &= \text{col rank}(A) \end{aligned}$$

Thus row rank$(A) \geq$ col rank(A). Hence row rank(A) = col rank(A), if $A \neq O_{m,n}$.
And if $A = O_{m,n}$, then row rank$(A) = 0 =$ col. rank(A).

Theorem 12 (Rank - Nullity Theorem)
Let A be any $m \times n$ matrix. Then
(a) row rank(A) + nullity(A) = no. of columns in $A = n$.
(b) col rank(A) + co-nullity(A) = no. of rows in $A = m$.

Proof. (a) Let $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_k\}$ be a basis of $Null(A)$. Then we can find $\{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r\}$ such that $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_k, \underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r\}$ is a basis of $\mathbb{R}^n$. Then nullity$(A) = k$. We claim that row rank$(A) = r$. We will show that $\{A\underset{\sim}{w}_1, \ldots, A\underset{\sim}{w}_r\}$ is a basis of $Col\,Sp(A)$

Let $\underset{\sim}{x} = \alpha_1 \underset{\sim}{u}_1 + \cdots + \alpha_k \underset{\sim}{u}_k + \beta_1 \underset{\sim}{w}_1 + \cdots + \beta_r \underset{\sim}{w}_r$. Then

$$\begin{aligned} A\underset{\sim}{x} &= \alpha_1 A\underset{\sim}{u}_1 + \cdots + \alpha_k A\underset{\sim}{u}_k + \beta_1 A\underset{\sim}{w}_1 + \cdots + \beta_r A\underset{\sim}{w}_r \\ &= \alpha_1 \underset{\sim}{0} + \cdots + \alpha_k \underset{\sim}{0} + \beta_1 (A\underset{\sim}{w}_1) + \cdots + \beta_r (A\underset{\sim}{w}_r) \\ &= \beta_1 (A\underset{\sim}{w}_1) + \cdots + \beta_k (A\underset{\sim}{w}_k). \end{aligned}$$

So $ColSp(A) = \{A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n\}$
$= \{\beta_1 A\underset{\sim}{w}_1 + \cdots + \beta_r A\underset{\sim}{w}_r : \beta_i \in \mathbb{R}\}$
$= \text{span}\{A\underset{\sim}{w}_1, \ldots, A\underset{\sim}{w}_r\}$.

Now suppose $c_1 A\underset{\sim}{w}_1 + \cdots + c_r A\underset{\sim}{w}_r = \underset{\sim}{0}$. Then
$A(c_1\underset{\sim}{w}_1 + \cdots + c_r\underset{\sim}{w}_r) = A(\underset{\sim}{0}) = \underset{\sim}{0}$.
So $c_1\underset{\sim}{w}_1 + \cdots + c_r\underset{\sim}{w}_r \in Null(A)$. Thus
$c_1\underset{\sim}{w}_1 + \cdots + c_r\underset{\sim}{w}_r = d_1\underset{\sim}{u}_1 + \cdots + d_k\underset{\sim}{u}_k$ for some $d_1, \ldots, d_k$ because $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_k\}$ is a basis of $Null(A)$.
So $(-d_1)\underset{\sim}{u}_1 + \cdots + (-d_k)\underset{\sim}{u}_k + c_1\underset{\sim}{w}_1 + \cdots + c_r\underset{\sim}{w}_r = \underset{\sim}{0}$.
Since $\{\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_k, \underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_r\}$ is a basis of $\mathbb{R}^n$, it follows that $-d_1 = \cdots = -d_k = c_1 = c_2 = \cdots = c_r = 0$.
So $\{A\underset{\sim}{w}_1, \ldots, A\underset{\sim}{w}_r\}$ is linearly independent. Hence $\{A\underset{\sim}{w}_1, \ldots, A\underset{\sim}{w}_r\}$ is a basis of $ColSp(A)$. So
row rank(A) = col rank(A) = r. Thus
row rank(A) + nullity(A) = $k + r$ = no. of col. in $A = n$

(b) This follows immediately by considering A^T.
col rank(A) + co-nullity(A) = row rank(A^T) + nullity(A^T)
= no. of columns in A^T
= no. of rows in $A = m$.

<u>Remark</u>: There is also another definition of rank. Let A be an $m \times n$ matrix. We define the <u>determinant rank</u> of A by

det rank(A) = size of the largest square sub-matrix of with non-zero determinant.

It can be shown that det rank(A) = row rank(A). So all three of the following row rank(A), col rank(A), det rank(A) can be just referred to as the <u>rank of A</u>.

§6. Finding bases for the four Fundamental subspaces

1. To find a basis of RowSp(A), we transform A into its *reduced row echelon form* A_{RR}. The non-zero rows of A_{RR} will form a basis of RowSp(A)
2. To find a basis of Null(A), we transform A_{RR} into the *supplemented square matrix* A_s by inserting or deleting rows of zeros so that we get a square matrix with the leading 1's in the diagonal. A basis of Null(A) consists of the nonzero columns of $(I - A_s)$.

Ex. 1. Find bases of RowSp(A) & Null(A) for the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 1 & 2 & -2 & 0 \\ -2 & -4 & 4 & 0 \end{bmatrix}$$

Sol.

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 1 & 2 & -2 & 0 \\ -2 & -4 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \quad \begin{matrix} R2 := R2 - R1 \\ R3 := R3 + 2R1 \end{matrix}$$

leading 1's underlined

$$\longrightarrow A_{RR} = \begin{bmatrix} \underline{1} & 2 & 0 & -4 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R1 := R1 + 3R2 \\ \\ R3 := R3 + 2R2 \end{matrix}$$

$$\longrightarrow A_s = \begin{bmatrix} \underline{1} & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ \leftarrow \text{insert zero row} \\ \\ \\ \end{matrix}$$

$$I - A_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) ∴ a basis of RowSp(A) = $\{(1, 2, 0, -4), (0, 0, 1, -2)\}$.

Ex. 1(b) Also a basis of Null(A) is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$.

To find a basis of ColSp(A) we transform $[A \mid I_n]$ into row echelon form with leading 1's $[U \mid E]$. Then

3. A basis of ColSp(A) will be the columns of A that corresponds to the columns of A_{RE} with leading 1's.

4. A basis of CoNull(A) will be the rows of E that corresponds to the zero rows of U

Ex. 2 Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 3 & 6 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. (a) Find a basis for ColSp(A)
(b) Find a basis for CoNull(A).

$$\left[\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 6 & -2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2:=R2+R1 \\ R3:=R3-3R1 \\ R4:=R4-R1 \end{array}$$

$\uparrow$ A $\uparrow$

columns of A corresponding to the columns of U with leading 1's

$$\to \left[\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R4:=R4-2R3 \end{array}$$

$$\to \left[\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ R2:=R3 \\ R3:=R2 \\ \\ \end{array}$$

$\underbrace{\qquad}_{U}$ $\underbrace{\qquad}_{E}$

(a) A basis of Col. Sp(A) $= \{(1,-1,3,1)^T, (-1,1,-2,1)^T\}$

(b) A basis of CoNull(A) $= \{(1,1,0,0), (5,0,-2,1)\}$.

Theorem 13 : Let A be an $m \times n$ matrix and B = a basis of CoNull(A), C = a basis of ColSp(A), D = a basis of RowSp(A) & E = a basis of Null(A).
If we let
[B] = matrix whose rows are the vectors in B
[C] = matrix whose columns are the vectors in C
[D] = matrix whose rows are the vectors in D
[E] = matrix whose columns are the vectors in E.
Then $[B][C] = O_{m-r,r}$ & $[D][E] = O_{r,n-r}$.
where r = row rank(A).

Ex. 3 Let $\underset{4\times 3}{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 3 & 6 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. Then $m=4$, $n=3$, and $r=2$.

$$\underset{2\times 4}{[B]}\ \underset{4\times 2}{[C]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 5 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{4-2,2}$$

Ex. 4 Let $\underset{3\times 4}{A} = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 1 & 2 & -2 & 0 \\ -2 & -4 & 4 & 0 \end{bmatrix}$. Then $m=3$, $n=2$, and $r=2$.

$$\underset{2\times 4}{[D]}\ \underset{4\times 2}{[E]} = \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2,4-2}$$

Ex. 5

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 3 & 6 & -2 \\ 1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R2 := R2 + R1 \\ R3 := R3 - 3R1 \\ R4 := R4 - R1 \end{matrix}$$

$$\longrightarrow A_s \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R1 := R2 + R3 \\ R4 := R4 - 2R3 \end{matrix}$$

a) A basis for $RowSp(A) = \{(1,2,0), (0,0,1)\}$

$$I - A_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) A basis for $Null(A) = \{(-2, 1, 0)^T\}$

(c) $\underset{2\times 3}{[D]}\ \underset{3\times 1}{[E]} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = O_{2,3-2}$

Ex 6

$$\left[\begin{array}{cccc|ccc} 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 & 0 & 1 & 0 \\ -2 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 4 & 2 & 0 & 1 \end{array}\right] \quad \begin{matrix} R2 := R2 - R1 \\ R3 := R3 + 2R1 \end{matrix}$$

$\uparrow$ (column 1) $\quad \uparrow$ (column 3)

$$\longrightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{array}\right] \quad R3 := R3 + 2R2$$

a) A basis for $CoNull(A) = \{[0\ 2\ 1]\}$

b) A basis for $ColSp(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} \right\}$

(c) $\underset{1\times 3}{[B]}\ \underset{3\times 2}{[C]} = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ -2 & 4 \end{bmatrix} = \underset{1\times 2}{\begin{bmatrix} 0 & 0 \end{bmatrix}} = O_{3-2,2}$