Ch.5 – Linear maps & their matrix representations

§1. Linear maps (or linear transformations)

Def. Let V & W be vector spaces. A <u>linear map</u> (or <u>linear transformation</u>) is any function $L: V \to W$ such that (a) $(\forall \underset{\sim}{x}, \underset{\sim}{y} \in V)\{L(\underset{\sim}{x}+\underset{\sim}{y}) = L(\underset{\sim}{x}) + L(\underset{\sim}{y})\}$ &
(b) $(\forall \alpha \in \mathbb{R})(\forall \underset{\sim}{x} \in V)\ \{L(\alpha \underset{\sim}{x}) = \alpha L(\underset{\sim}{x})\}$.

Ex. 1 Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_3 \\ 3x_2 - x_3 \end{pmatrix}$.

Then $L\left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right\} = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} (x_1+y_1) + 2(x_3+y_3) \\ 3(x_2+y_2) - (x_3+y_3) \end{pmatrix}$

$$= \begin{pmatrix} x_1 + 2x_3 \\ 3x_2 - x_3 \end{pmatrix} + \begin{pmatrix} y_1 + 2y_3 \\ 3y_2 - y_3 \end{pmatrix} = L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + L\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

& $L\left\{\alpha\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right\} = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + 2\alpha x_3 \\ 3\alpha x_2 - \alpha x_3 \end{pmatrix} = \alpha\begin{pmatrix} x_1 + 2x_3 \\ 3x_2 - x_3 \end{pmatrix} = \alpha L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

So L is a linear map.

Ex. 2 Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^3 \\ x_1 x_2 \end{pmatrix}$.

Then $L\left\{\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\} = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \begin{pmatrix} (\alpha x_1)^3 \\ (\alpha x_1)(\alpha x_2) \end{pmatrix} = \begin{pmatrix} \alpha^3 x_1^3 \\ \alpha^2 x_1 x_2 \end{pmatrix} \neq \alpha L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

if we take $\alpha = 2$. So L is not a linear map.

<u>Prop. 1</u> If $L: V \to W$ is a linear map, then

(a) $L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$

(b) $L(-\underset{\sim}{x}) = -L(\underset{\sim}{x})$

(c) $L(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}) = \alpha L(\underset{\sim}{x}) + \beta L(\underset{\sim}{y})$

Proof: (a) $L(\underset{\sim}{0}_V) = L(0.\underset{\sim}{0}_V) = 0.L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$

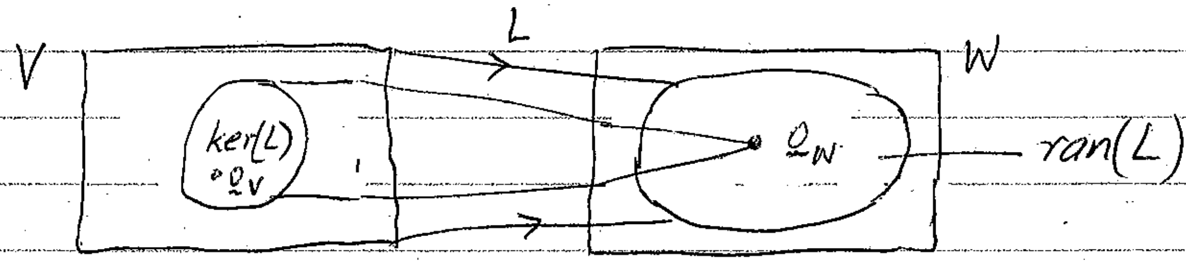
(b) $L(-\underset{\sim}{x}) = L((-1).\underset{\sim}{x}) = (-1).L(\underset{\sim}{x}) = -L(\underset{\sim}{x})$.

(c) $L(\alpha\underset{\sim}{x} + \beta\underset{\sim}{y}) = L(\alpha\underset{\sim}{x}) + L(\beta\underset{\sim}{y}) = \alpha L(\underset{\sim}{x}) + \beta L(\underset{\sim}{y})$.

Def. Let $L: V \to W$ be a linear map. We define the kernel of L by $\ker(L) = \{\underset{\sim}{x} \in V : L(\underset{\sim}{x}) = \underset{\sim}{0}_W\}$.

Also let S be a subspace of V. We define the image of S under L by $L[S] = \{L(\underset{\sim}{x}) : \underset{\sim}{x} \in S\}$.
If $S = V$, we call $L[V]$ the range of L and denote it by $\text{ran}(L)$.

Finally let R be a subspace of W. We define the pre-image of R under L by $L^{-1}[R] = \{\underset{\sim}{x} \in V : L(\underset{\sim}{x}) \in R\}$.
Note that $\ker(L) = \{x \in V : L(\underset{\sim}{x}) = \underset{\sim}{0}_W\} = L^{-1}[\underset{\sim}{0}_W]$.



Ex.3 Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_2 - x_3 \end{pmatrix}$.
Then L is a linear map.

(a) $\ker(L) = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 + x_3 \\ x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

$= \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = -x_3 \,\&\, x_3 = x_2\right\} = \left\{\begin{pmatrix} -\alpha \\ \alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R}\right\} = \left\{\alpha\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R}\right.$

Ex. 3 (b) $\operatorname{ran}(L) = \left\{ L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V \right\} = \left\{ \begin{pmatrix} x_1 + x_3 \\ x_2 - x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$

$= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} : x_i \in \mathbb{R} \right\}$

$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2.$

(c) Let $S = \left\{ \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$. Then $S = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$.

$L[S] = \left\{ L\begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \alpha + \alpha \\ 0 - \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \left\{ \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$

(d) Let $R = \left\{ \begin{pmatrix} 3\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$. Then $R = \left\{ \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$ is subspace of $\mathbb{R}^2$. Also

$L^{-1}[R] = \{ \underset{\sim}{x} \in \mathbb{R}^3 : L(\underset{\sim}{x}) \in R \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{array}{l} x_1 + x_3 = 3\alpha \\ x_2 - x_3 = \alpha \end{array} \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = 3\alpha - x_2 \ \& \ x_3 = x_2 - \alpha \right\}$

$= \left\{ \begin{pmatrix} 3\alpha - \beta \\ \beta \\ \beta - \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} = \left\{ \alpha \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

<u>Proposition 2</u>. Let $L : V \to W$ be a linear map, S be a subspace of V, and R be a subspace of W. Then

(a) $\ker(L)$ & $L^{-1}[R]$ are subspaces of V.

(b) $\operatorname{ran}(L)$ & $L[S]$ are subspaces of W.

<u>Proof</u>: (a) Since $\ker(L) = L^{-1}[\{\underset{\sim}{0}_W\}]$, it will suffice to prove the result only for $L^{-1}[R]$. Now $L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W \in R$ because R is a subspace of W. So $\underset{\sim}{0}_V \in L^{-1}[R]$. Hence $L^{-1}[R] \neq \emptyset$. Suppose $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in L^{-1}[R]$ and $\alpha \in \mathbb{R}$. Then $L(\underset{\sim}{x}_1) \in R$ & $L(\underset{\sim}{x}_2) \in R$. So

$L(\underset{\sim}{x}_1+\underset{\sim}{x}_2) = L(\underset{\sim}{x}_1) + L(\underset{\sim}{x}_2) \in R$ bec. R is a subspace
& $L(\alpha \underset{\sim}{x}_1) = \alpha L(x_1) \in R$ because R is a subspace.
Hence $x_1+x_2 \in L^{-1}[R]$ and $\alpha \underset{\sim}{x}_1 \in L^{-1}[R]$. Thus $L^{-1}[R]$ is a subspace of V

(b) Since $\text{ran}(L) = L[V]$, it will suffice to prove the result only for $L[S]$. Now for any subspace S of V, $\underset{\sim}{0}_V \in S$ & $L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$. So $\underset{\sim}{0}_W \in L[S]$. Hence $L[S] \neq \emptyset$. Also if $\underset{\sim}{y}_1, \underset{\sim}{y}_2 \in L[S]$ & $\alpha \in \mathbb{R}$, then we can find $x_1, x_2 \in S$ such that $y_1 = L(\underset{\sim}{x}_1)$ & $y_2 = L(\underset{\sim}{x}_2)$. So

$$\underset{\sim}{y}_1 + \underset{\sim}{y}_2 = L(\underset{\sim}{x}_1) + L(\underset{\sim}{x}_2) = L(\underset{\sim}{x}_1+\underset{\sim}{x}_2) \in L[S] \text{ and}$$
$$\alpha \underset{\sim}{y}_1 = \alpha L(x_1) = L(\alpha \underset{\sim}{x}_1) \in L[S] \text{ because}$$

$\underset{\sim}{x}_1 + \underset{\sim}{x}_2 \in S$ & $\alpha \underset{\sim}{x}_1 \in S$, since S is a subspace of V
$\therefore$ $L[S]$ is a subspace of W.

<u>Prop. 3</u> Let A be an $m \times n$ matrix. Define $L_A : \mathbb{R}^n \to \mathbb{R}^m$ by $L_A(\underset{\sim}{x}) = A\underset{\sim}{x}$. Then L_A is a linear map.

<u>Proof</u>: Let $\alpha \in \mathbb{R}$ and $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$. Then
$L_A(\underset{\sim}{x}+\underset{\sim}{y}) = A(\underset{\sim}{x}+\underset{\sim}{y}) = A\underset{\sim}{x} + A\underset{\sim}{y} = L_A(\underset{\sim}{x}) + L_A(\underset{\sim}{y})$
& $L(\alpha \underset{\sim}{x}) = A(\alpha \underset{\sim}{x}) = \alpha(A\underset{\sim}{x}) = \alpha L_A(\underset{\sim}{x})$. Hence L_A is a linear map.

<u>Ex. 4</u> Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}$. Then $L_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
$= \begin{bmatrix} 2x_1 + x_3 \\ x_2 - 3x_3 \end{bmatrix}$. We can check directly that $L_A : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map or we can just appeal to Proposition 3.

Recall that a straight line in 2 or 3 dimensions can be specified by a fixed vector in the line & by a vector giving the direction of the line,

$$\Lambda = \{\underset{\sim}{a} + \alpha \underset{\sim}{b} : \alpha \in \mathbb{R}\}$$

Recall also that a plane in 3 dimensions can be specified by a fixed vector in the plane & two independent vectors giving the parallel plane through the origin. $\Pi = \{\underline{a} + \alpha \underline{b} + \beta \underline{c} : \alpha, \beta \in \mathbb{R}\}$

Fact 5: If $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map, and Λ & Π are a line & a plane, then $L[\Lambda]$ & $L[\Pi]$ are also a line & a plane (or degenerate ones)

Proof: $L[\Lambda] = \{L(\underline{a} + \alpha \underline{b}) : \alpha \in \mathbb{R}\} = \{L(\underline{a}) + L(\alpha \underline{b}) : \alpha \in \mathbb{R}\}$
$= \{L(\underline{a}) + \alpha L(\underline{b}) : \alpha \in \mathbb{R}\} =$ a straight line.

$L[\Pi] = \{L(\underline{a} + \alpha \underline{b} + \beta \underline{c}) : \alpha, \beta \in \mathbb{R}\} = \{L(\underline{a}) + L(\alpha \underline{b}) + L(\beta \underline{c}) : \alpha, \beta \in \mathbb{R}\}$
$= \{L(\underline{a}) + \alpha L(\underline{b}) + \beta L(\underline{c}) : \alpha, \beta \in \mathbb{R}\} =$ a plane.

If $L(\underline{b}) = \underline{0}$, then $L[\Lambda] =$ a point $=$ a degenerate line.
If $L(\underline{b})$ or $L(\underline{c}) = \underline{0}$, then $L[\Pi] =$ a line $=$ a degenerate plane.
If $L(\underline{b})$ & $L(\underline{c}) = \underline{0}$, then $L[\Pi] =$ a point $=$ a degenerate plane.

Def. Let $\underline{u}, \underline{v}$ & $\underline{w}$ be vectors in $\mathbb{R}^3$. The parallelogram determined by $\underline{u}$ & $\underline{v}$ is defined by $P(\underline{u}, \underline{v}) = \{\alpha \underline{u} + \beta \underline{v} : 0 \leq \alpha \leq 1 \;\&\; 0 \leq \beta \leq 1\}$. The parallelopiped determined by $\underline{u}, \underline{v}$ & $\underline{w}$ is defined by $S(\underline{u}, \underline{v}, \underline{w}) = \{\alpha \underline{u} + \beta \underline{v} + \gamma \underline{w} : 0 \leq \alpha, \beta, \gamma \leq 1\}$.

Fact 6: If $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and P & S are as above, then $L[P]$ & $L[S]$ are also parallelogram & parallelopiped.

§2. The matrix representation of a lin. map w.r.t. the standard base.

Ex 1 Consider the linear map $L: \mathbb{R}^3 \to \mathbb{R}^2$ which is defined by $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_3 \\ 3x_2 - x_3 \end{pmatrix}$. Is there a 2×3 matrix A such that $L\underset{\sim}{x} = A\underset{\sim}{x}$?

Sol. Yes. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and suppose

$L\underset{\sim}{x} = A\underset{\sim}{x}$. Then $x_1 + 0x_2 + 2x_3 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$

$0x_1 + 3x_2 - x_3 = a_{21}x_1 + a_{22}x_2 + a_{13}x_3$

So if we take $a_{11} = 1$, $a_{12} = 0$, $a_{13} = 2$ and

$a_{21} = 0$, $a_{22} = 3$, $a_{23} = -1$,

then $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \end{bmatrix}$ & $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} x_1 + 2x_3 \\ 3x_2 - x_3 \end{pmatrix}$.

<u>Theorem 7</u> (Matrix Representation Theorem w.r.t. standard bases)

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $L(\underset{\sim}{x}) = A\underset{\sim}{x}$ where A is the $m \times n$ matrix $[L(\underset{\sim}{e}_1)\ L(\underset{\sim}{e}_2)\ \cdots\ L(\underset{\sim}{e}_n)]$, i.e., $L(\underset{\sim}{e}_j) =$ the j-th column of A. Denote A by ${}_{E_m}(L)_{E_n}$.

<u>Proof</u>: Recall that $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underset{\sim}{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, ..., & $\underset{\sim}{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

So $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2 + \cdots + x_n\underset{\sim}{e}_n$

$\therefore L(\underset{\sim}{x}) = L(x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2 + \cdots + x_n\underset{\sim}{e}_n)$

$= x_1 L(\underset{\sim}{e}_1) + x_2 L(\underset{\sim}{e}_2) + \cdots + x_n L(\underset{\sim}{e}_n)$

$= [L(\underset{\sim}{e}_1)\ \ L(\underset{\sim}{e}_2)\ \cdots\ L(\underset{\sim}{e}_n)]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\underset{\sim}{x}$.

Recall that ${}_{E_m}(L)_{E_n}$ denotes the matrix rep. of $L: \mathbb{R}^n \to \mathbb{R}^m$ w.r.t E_n & E_m

Theorem 8: Let $L_1: \mathbb{R}^n \to \mathbb{R}^m$ & $L_2: \mathbb{R}^p \to \mathbb{R}^n$ be linear maps. Then (a) $L_1 \circ L_2: \mathbb{R}^p \to \mathbb{R}^m$ is a linear map &

(b) $${}_{E_m}(L_1 \circ L_2)_{E_p} = {}_{E_m}(L_1)_{E_n} \; {}_{E_n}(L_2)_{E_p}.$$

Proof (a) Do for H.W.

(b) Let $A = [a_{ij}] = {}_{E_m}(L_1 \circ L_2)_{E_p}$, $B = [b_{ij}] = {}_{E_m}(L_1)_{E_n}$ and $C = [c_{ij}] = {}_{E_n}(L_2)_{E_p}$. Then for any $j = 1, \ldots, p$

$$\text{column } j \text{ of } A = (L_1 \circ L_2)(\underset{\sim}{e}_j) = L_1(L_2(\underset{\sim}{e}_j))$$

$$= L_1(c_{1j}\underset{\sim}{e}_1 + c_{2j}\underset{\sim}{e}_2 + \cdots + c_{nj}\underset{\sim}{e}_n)$$

$$= c_{1j} L_1(\underset{\sim}{e}_1) + c_{2j} L_1(\underset{\sim}{e}_2) + \cdots + c_{nj} L_1(e_n)$$

$$\begin{aligned}
= \; & c_{1j}(b_{11}e_1 + b_{21}\underset{\sim}{e}_2 + \cdots + b_{m1}\underset{\sim}{e}_m) \\
& + c_{2j}(b_{12}\underset{\sim}{e}_1 + b_{22}\underset{\sim}{e}_2 + \cdots + b_{m2}\underset{\sim}{e}_m) \\
& \;\;\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& + c_{nj}(b_{1n}\underset{\sim}{e}_1 + b_{2n}\underset{\sim}{e}_2 + \cdots + b_{mn}\underset{\sim}{e}_m)
\end{aligned}$$

$$\begin{aligned}
= \; & (b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1n}c_{nj})\underset{\sim}{e}_1 \\
& + (b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2n}c_{nj})\underset{\sim}{e}_2 \\
& \;\;\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& + (b_{m1}c_{1j} + b_{m2}c_{2j} + \cdots + b_{mn}c_{nj})\underset{\sim}{e}_m
\end{aligned}$$

$$= \begin{bmatrix} (\text{row 1 of } B) \cdot (\text{column } j \text{ of } C) \\ (\text{row 2 of } B) \cdot (\text{column } j \text{ of } C) \\ \vdots \qquad\qquad \vdots \\ (\text{row } m \text{ of } B \cdot (\text{column } j \text{ of } C) \end{bmatrix} = \text{column } j \text{ of } (BC)$$

$\therefore$ column j of A = column j of (BC) for $j = 1, \ldots, p$.

$\therefore$ $A = BC$. Hence ${}_{E_m}(L_1 \circ L_2)_{E_p} = {}_{E_m}(L_1)_{E_n} \, {}_{E_n}(L_2)_{E_p}$.

Ex2 Find the matrix representation w.r.t. the standard basis of the following linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) reflection in the x_1-axis, i.e., about the polar line $\theta = 0$

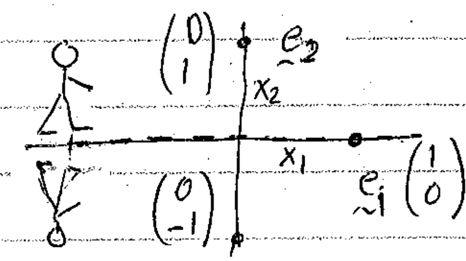
(b) reflection in the line $x_1 + x_2 = 0$, i.e., about line $\theta = \frac{3\pi}{4}$

(c) rotation through $\pi/2$ radians about the origin

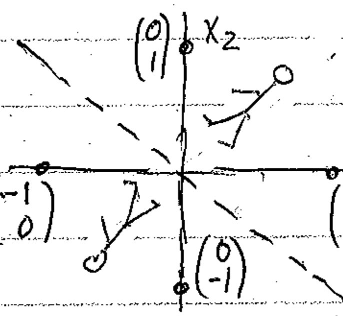
(d) rotation through θ radians about the origin

(e) reflection in the line through the origin that makes an angle of θ radians with the x_1-axis.

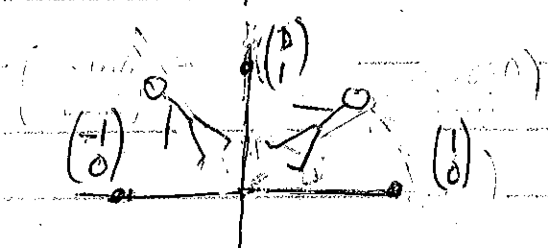
(a) $Ref(0) = [L_a(\underset{\sim}{e}_1) \; L_a(\underset{\sim}{e}_2)] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



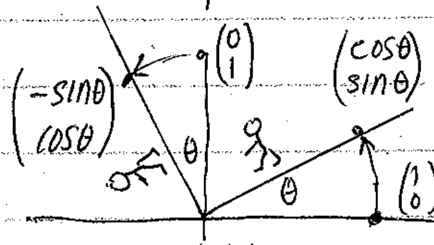
(b) $Ref(\frac{3\pi}{4}) = [L_b(\underset{\sim}{e}_1) \; L_b(\underset{\sim}{e}_2)] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$



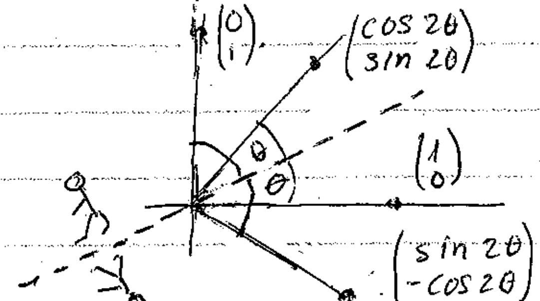
(c) $Rot(\frac{\pi}{2}) = [L_c(\underset{\sim}{e}_1) \; L_c(\underset{\sim}{e}_2)] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



(d) $Rot(\theta) = [L_d(\underset{\sim}{e}_1) \; L_d(\underset{\sim}{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



(e) $Ref(\theta) = [L_e(\underset{\sim}{e}_1) \; L_e(\underset{\sim}{e}_2)] = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$



Observe that all five of these matrices are orthogonal and that the determinant of any reflection is -1, while the determinant of any rotation is 1. Recall that a matrix A is orthogonal if $A^T A = I$.

Notice also that by using matrix multiplication we have

a) $Rot(\alpha)\, Rot(\beta) = Rot(\alpha+\beta)$

b) $Ref(\alpha)\, Ref(\beta) = Rot(2(\alpha-\beta))$

c) $Rot(\alpha)\, Ref(\beta) = Ref((\alpha/2)+\beta)$

d) $Ref(\alpha)\, Rot(\beta) = Ref(\alpha - (\beta/2))$

It will therefore come as no surprise that

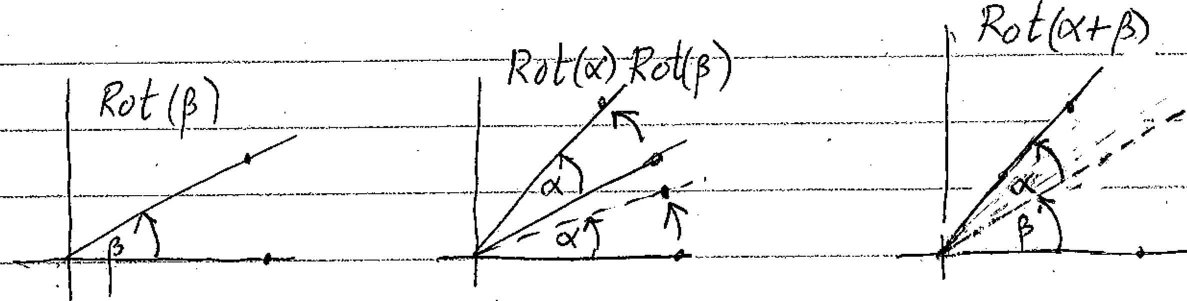
e) $Ref(\alpha)\, Ref(\alpha) = I = Rot(0)$

f) $Rot(\alpha)\, Rot(-\alpha) = I = Rot(0)$

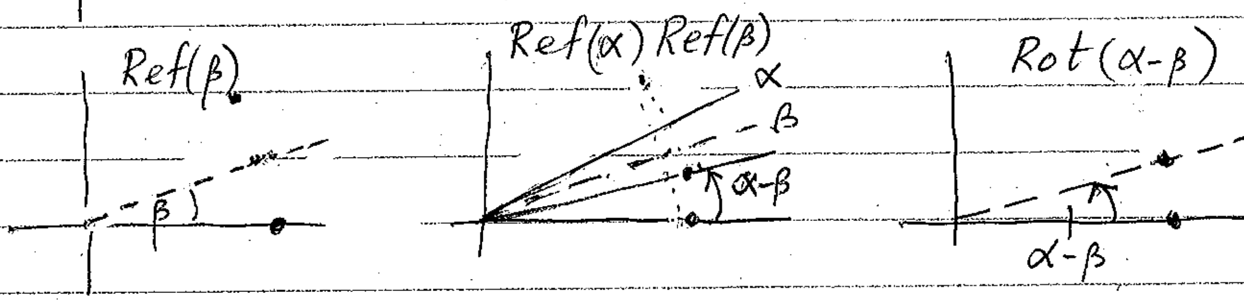
g) $Ref(0)\, Ref(\pi/2) = Rot(-\pi) = Rot(\pi)$

h) $Ref(\pi/2)\, Ref(0) = Rot(\pi)$

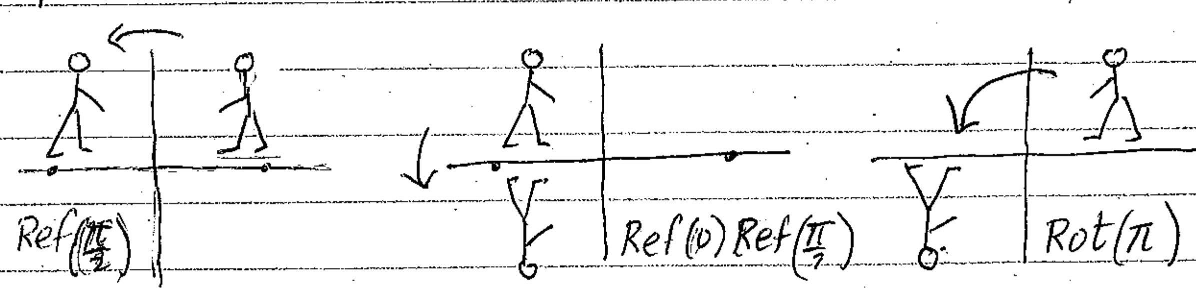
a)



b)



g)



§3. Matrix Rep. w.r.t arbitrary bases & change of bases

Def. Let V be a vector space and $B = \langle \underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_k \rangle$ be an ordered basis of V. If $\underset{\sim}{u} \in V$, then $\underset{\sim}{u}$ can be uniquely written as

$$\underset{\sim}{u} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k \quad \text{with } c_i \in \mathbb{R}.$$

The terms of $\langle c_1, \ldots, c_k \rangle$ are called the <u>coordinates</u> of $\underset{\sim}{u}$ w.r.t. the ordered basis B and we use the notation $[\underset{\sim}{u}]_B$ to denote the column vector $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$.

Ex.1 Let $B = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$. Then B is an ordered basis of $\mathbb{R}^2$. The vector $\underset{\sim}{u} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ can be uniquely written as $u = \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. So the seq. of coordinates of $\underset{\sim}{u}$ w.r.t. the ordered basis B is $\langle 5, 2 \rangle$ and we write $[\underset{\sim}{u}]_B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

The seq. of coordinates of $\underset{\sim}{u}$ w.r.t. the standard basis $E_2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$ of $\mathbb{R}^2$ are, of course, $\langle 7, 3 \rangle$ because $u = \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So $[\underset{\sim}{u}]_{E_n} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

<u>Theorem 9</u> (Matrix Rep. Theorem w.r.t. arb. bases)
Let $B = \langle \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n \rangle$ & $D = \langle \underset{\sim}{w}_1, \ldots, w_m \rangle$ be ordered bases of the vector spaces V & W, respectively. Then for each linear transformation $L: V \to W$,

$$[L(\underset{\sim}{x})]_D = {}_D(L)_B \, [\underset{\sim}{x}]_B,$$

where

$${}_D(L)_B = \left[[L(\underset{\sim}{v}_1)]_D, \ [L(\underset{\sim}{v}_2)]_D, \ \ldots, \ [L(\underset{\sim}{v}_n)]_D \right]$$

Note that ${}_D(L)_B$ will be an $m \times n$ matrix. The proof is very similar to that of Theorem 7.

Theorem 10: Let $L_1 : V \to W$ & $L_2 : U \to V$ be linear maps & $F = \langle \underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_p \rangle$, $G = \langle \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n \rangle$ & $H = \langle \underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_m \rangle$ be ordered bases of U, V & W. Then

$$_H(L_1 \circ L_2)_F = {}_H(L_1)_G \; {}_G(L_2)_F .$$

Proof: Let $_H(L_1 \circ L_2)_F = A = [a_{ij}]$, $_H(L_1)_G = B = [b_{ij}]$ and $_G(L_2)_F = C = [c_{ij}]$. Then

$$(L_1 \circ L_2)(\underset{\sim}{u}_j) = L_1(L_2(\underset{\sim}{u}_j))$$

$$= L_1(c_{1j} \underset{\sim}{v}_1 + c_{2j} \underset{\sim}{v}_2 + \cdots + c_{nj} \underset{\sim}{v}_n) \quad \text{by def. of } C$$

$$= c_{1j} L_1(\underset{\sim}{v}_1) + c_{2j} L_1(\underset{\sim}{v}_2) + \cdots + c_{nj} L_1(\underset{\sim}{v}_n)$$

$$\begin{aligned} = \; & c_{1j}(b_{11} \underset{\sim}{w}_1 + b_{21} \underset{\sim}{w}_2 + \cdots + b_{m1} \underset{\sim}{w}_m) && \text{by def. of } B \\ + \; & c_{2j}(b_{12} \underset{\sim}{w}_1 + b_{22} \underset{\sim}{w}_2 + \cdots + b_{m2} \underset{\sim}{w}_m) && \text{"} \\ & \vdots \hspace{6em} \vdots && \vdots \\ + \; & c_{nj}(b_{1n} \underset{\sim}{w}_1 + b_{2n} \underset{\sim}{w}_2 + \cdots + b_{mn} \underset{\sim}{w}_m) && \text{by def. of } B \end{aligned}$$

$$\begin{aligned} = \; & (b_{11} c_{1j} + b_{12} c_{2j} + \cdots + b_{1n} c_{nj}) \underset{\sim}{w}_1 \\ + \; & (b_{21} c_{1j} + b_{22} c_{2j} + \cdots + b_{2n} c_{nj}) \underset{\sim}{w}_2 \\ & \vdots \hspace{14em} \vdots \\ + \; & (b_{m1} c_{1j} + b_{m2} c_{2j} + \cdots + b_{mn} c_{nj}) \underset{\sim}{w}_m \end{aligned}$$

$$\begin{aligned} = \; & (BC)[1,j] \; \underset{\sim}{w}_1 \\ + \; & (BC)[2,j] \; \underset{\sim}{w}_2 \\ & \vdots \\ + \; & (BC)[m,j] \; \underset{\sim}{w}_m \end{aligned} \qquad \text{. But } (L_1 \circ L_2)(\underset{\sim}{u}_j) = \begin{aligned} & A[1,j] \; \underset{\sim}{w}_1 \\ + \; & A[2,j] \; \underset{\sim}{w}_2 \\ & \vdots \\ + \; & A[m,j] \; \underset{\sim}{w}_m \end{aligned}$$

by definition of A.

So column j of BC = column j of A. Hence $A = BC$. Thus $_H(L_1 \circ L_2)_F = {}_H(L_1)_G \; {}_G(L_2)_F$.

Ex. 2 Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 - 2x_2 \\ x_1 + x_2 \end{pmatrix}$ and $B = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = \langle \underset{\sim}{v_1}, \underset{\sim}{v_2} \rangle$. Find

(a) ${}_{E_2}(L)_{E_2}$ and (b) ${}_{E_2}(L)_B$ (c) ${}_B(L)_B$

Sol. (a) ${}_{E_2}(L)_{E_2} = \left[L\begin{pmatrix} 1 \\ 0 \end{pmatrix}, L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}.$

(b) $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4(1) - 2(1) \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

$L\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4(2) - 2(1) \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 6 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

So $[L(\underset{\sim}{v_1})]_{E_2} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ & $[L(\underset{\sim}{v_2})]_{E_2} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ $\therefore {}_{E_2}(L)_B = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$

(c) $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$L\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So $[L(\underset{\sim}{v_1})]_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ & $[L(\underset{\sim}{v_2})]_B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. $\therefore {}_B(L)_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$

Ex. 3 Let $X = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ (a) Find $[\underset{\sim}{X}]_B$ where $B = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$ and check that

(b) $[L(\underset{\sim}{X})]_{E_2} = {}_{E_2}(L)_B \, [\underset{\sim}{X}]_B$ & (c) $[L(X)]_B = {}_B(L)_B \, [\underset{\sim}{X}]_B$.

Sol. (a) $\underset{\sim}{X} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} = (-4)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (5)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So $[\underset{\sim}{X}]_B = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$.

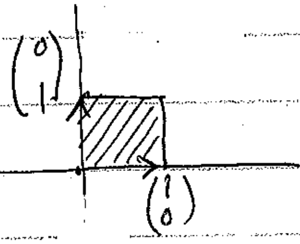
(b) ${}_{E_2}(L)_B \, [\underset{\sim}{X}]_B = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}$ and

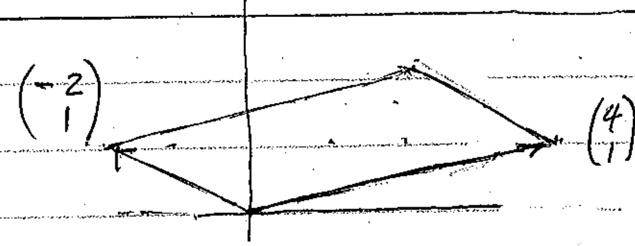
$[L(\underset{\sim}{X})]_{E_2} = \begin{bmatrix} 4(6) - 2(1) \\ 6 + 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix} = {}_{E_2}(L)_B \, [\underset{\sim}{X}]_B$

(c) $L(\underset{\sim}{X}) = \begin{pmatrix} 22 \\ 7 \end{pmatrix} = (-8)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 15\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Also ${}_B(L)_B \, [\underset{\sim}{X}]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -8 \\ 15 \end{pmatrix}$.

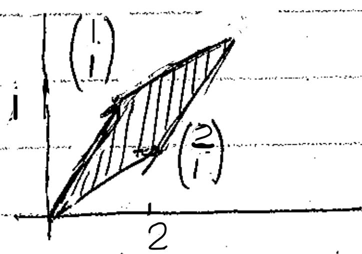
So $[L(\underset{\sim}{X})]_B = \begin{pmatrix} -8 \\ 15 \end{pmatrix} = {}_B(L)_B \, [\underset{\sim}{X}]_B$.

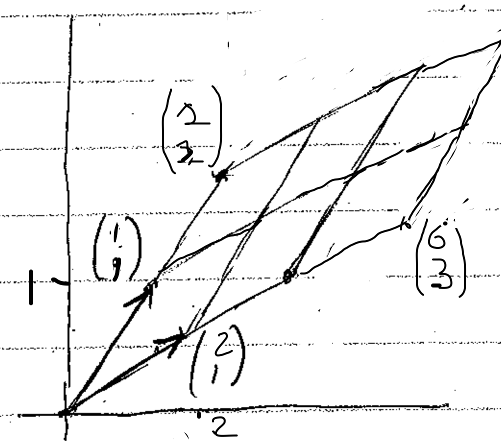
Ex 4 L w.r.t. the basis E_2





L w.r.t. the basis B.





Change of Basis

Let V be a vector space and suppose that $B = \langle \underset{\sim}{v_1}, \dots, \underset{\sim}{v_k} \rangle$ and $\mathcal{D} = \langle \underset{\sim}{w_1}, \dots, w_k \rangle$ are ordered bases of V. Given $[\underset{\sim}{x}]_B$, how can we find $[\underset{\sim}{x}]_D$?

Corollary 11 (Change of Basis Theorem)

Let B & D be ordered bases of the vector space V and $I: V \to V$ be the identity linear map (i.e., $I(\underset{\sim}{x}) = \underset{\sim}{x}$). Then $[\underset{\sim}{x}]_D = \{{}_D(I)_B\}[\underset{\sim}{x}]_B$ & $[\underset{\sim}{x}]_B = \{{}_D(I)_B\}^{-1}[\underset{\sim}{x}]_D$

Proof: This follows immediately from Theorem 9 because $[\underset{\sim}{x}]_D = [I(\underset{\sim}{x})]_D = \{{}_D(I)_B\}[\underset{\sim}{x}]_B$. Also $[\underset{\sim}{x}]_B = \{I_n\}[\underset{\sim}{x}]_B = \{{}_B(I)_B\}[\underset{\sim}{x}]_B$ $= \{{}_B(I \circ I)_B\}[\underset{\sim}{x}]_B = \{{}_B(I)_D\}\{{}_D(I)_B\}[\underset{\sim}{x}_B]$. So

$$\{{}_B(I)_D\}\{{}_D(I)_B\} = I_n, \text{ the } n \times n \text{ identity matrix.}$$

So ${}_D(I)_B$ is invertible and $\{{}_D(I)_B\}^{-1} = {}_B(I)_D$

Thus

$$[\underset{\sim}{x}]_B = \{{}_B(I)_D\}[\underset{\sim}{x}]_D = \{{}_D(I)_B\}^{-1}[\underset{\sim}{x}]_D.$$

Def. The transition matrix from the basis B to the basis D is defined to any $n\times n$ matrix S such that $[\underset{\sim}{x}]_D = (S)[\underset{\sim}{x}]_B$. So $S = {}_D(I)_B$.

Prop. 12 Let $B = \langle \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n \rangle$ be any ordered bases of $\mathbb{R}^n$ and $E_n = \langle \underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n \rangle$ be the standard basis of $\mathbb{R}^n$. Then ${}_B(I)_{E_n}$ is the $n\times n$ matrix with $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$ as its columns, i.e., ${}_B(I)_{E_n} = [\underset{\sim}{v}_1 \ \underset{\sim}{v}_2 \ \cdots \ \underset{\sim}{v}_n]$

Proof: If $\underset{\sim}{v}_j = a_{1j}\underset{\sim}{e}_1 + a_{2j}\underset{\sim}{e}_2 + \dots + a_{nj}\underset{\sim}{e}_n$, then $[I(\underset{\sim}{v}_j)]_{E_n} = [\underset{\sim}{v}_j]_{E_n} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$. So column j of ${}_B(I)_{E_n} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$

$$\therefore {}_{E_n}(I)_B = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [\underset{\sim}{v}_1 \ \underset{\sim}{v}_2 \ \cdots \ \underset{\sim}{v}_n].$$

Ex. 5 Let $B = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \end{bmatrix} \right\rangle$. Then ${}_{E_2}(I)_B = \begin{bmatrix} 2 & -7 \\ 1 & -3 \end{bmatrix}$

and ${}_B(I)_{E_2} = \{ {}_{E_2}(I)_B \}^{-1} = \begin{bmatrix} 2 & -7 \\ 1 & -3 \end{bmatrix}^{-1}$

$$= \frac{1}{(-6)-(-7)} \begin{bmatrix} -3 & 7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ -1 & 2 \end{bmatrix}$$

Note: A quick way to find $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ is to use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Ex. 6 Let $B = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\rangle$ and $D = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$. Find the following transition matrices.

(a) ${}_{E_2}(I)_B$ & ${}_D(I)_{E_2}$ and (b) ${}_D(I)_B$ & ${}_B(I)_D$

Sol. (a) ${}_{E_2}(I)_B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. Also ${}_{E_2}(I)_D = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

So ${}_D(I)_{E_2} = \{{}_{E_2}(I)_D\}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

(b) ${}_D(I)_B = {}_D(I)_{E_2}\,{}_{E_2}(I)_B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ -3 & -7 \end{bmatrix}$

$\therefore$ ${}_B(I)_D = \{{}_D(I)_B\}^{-1} = \begin{bmatrix} 5 & 12 \\ -3 & -7 \end{bmatrix}^{-1} = \frac{1}{-35+36} \begin{bmatrix} -7 & -12 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 3 & 5 \end{bmatrix}$.

Recall that an $n \times n$ matrix A was <u>similar</u> to the $n \times n$ matrix B if we can an invertible matrix P such that $A = P^{-1}BP$.

<u>Theorem 13</u>: The matrix A is similar to $B \Leftrightarrow A$ & B represent the same linear map $L: \mathbb{R}^n \to \mathbb{R}^n$ w.r.t. to two ordered bases F & G.

<u>Proof:</u> ($\Leftarrow$). Suppose $A = {}_G(L)_G$ & $B = {}_F(L)_F$. Then

$A = {}_G(L)_G = {}_G(I)_F\,{}_F(L)_F\,{}_F(I)_G = \{{}_G(I)_F\}\, B\, \{{}_F(I)_G\}$.

If we put $P = {}_F(I)_G$, we get $A = P^{-1}BP$.

($\Rightarrow$) Suppose A is similar to B. Then we can find an invertible matrix P such that $A = P^{-1}BP$. Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be defined by $L(\underset{\sim}{x}) = B\underset{\sim}{x}$. Then ${}_{E_n}(L)_{E_n} = B$. Let G be the basis consisting of the columns of P. Then $P = {}_{E_n}(I)_G$. So

${}_G(L)_G = {}_G(I)_{E_n}\,{}_{E_n}(L)_{E_n}\,{}_{E_n}(I)_G = P^{-1}BP = A$.