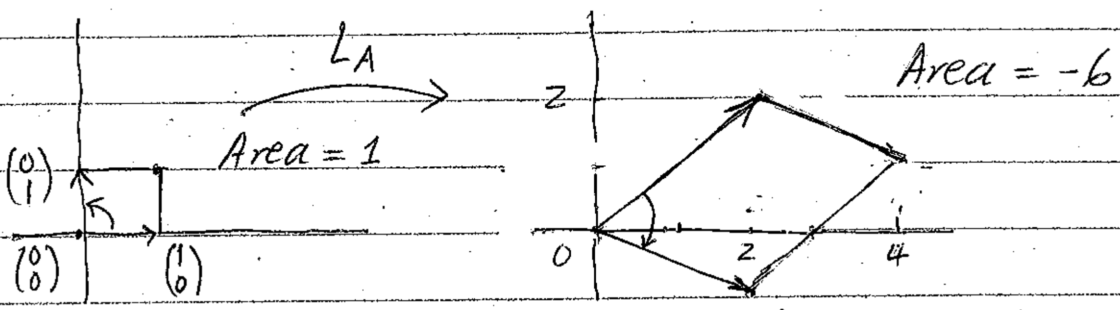
Ch 7 - Simplified representations of linear maps

§1. Eigenvalues & their corresponding eigenvectors

Consider the matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$. We know that A determines a linear map $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ that is defined by $L_A(\underset{\sim}{x}) = A\underset{\sim}{x}$. We can also get an idea of what L_A does by looking at the image of the unit square in $\mathbb{R}^2$.



From the image, we can see that $\det(A) = -6$. The linear map L_A has many different representation and some of these representations can be obtained by considering the matrices $P^{-1}AP$ where P is an invertible matrix. In order to find the simplest representation of L_A, we will introduce the concepts of eigenvalues and eigenvectors of A. Before we do this we need to extend the range of possible matrices P by discussing vector spaces over $\mathbb{C}$.

Def. Let $\mathbb{C}$ = the set of complex numbers. A vector space over $\mathbb{C}$ is defined in the same way as a vector space over $\mathbb{R}$, except that the scalars are now from the field $\mathbb{C}$ (instead of $\mathbb{R}$). An $m \times n$ matrix over $\mathbb{C}$ is one with entries from $\mathbb{C}$. We use $\mathbb{C}^{m\times n}$ to denote the set of all complex $m\times n$ matrice

Def. Let $z = a+ib$ be ~~rea~~ a complex number. The complex conjugate of z is defined by $\bar{z} = a-ib$.
If $\underset{\sim}{v} \in \mathbb{C}^n$ is a complex vector, we define the length of v by $\|\underset{\sim}{v}\| = \sqrt{\{\underset{\sim}{v}^T \bar{\underset{\sim}{v}}\}}$ where $\bar{\underset{\sim}{v}} = \begin{pmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{pmatrix}$. We also define $\underset{\sim}{v} \cdot \underset{\sim}{w} = \underset{\sim}{v}^T \bar{\underset{\sim}{w}}$.
Note that $\underset{\sim}{w} \cdot \underset{\sim}{v} = \underset{\sim}{w}^T \bar{\underset{\sim}{v}} = \bar{\underset{\sim}{v}}^T \underset{\sim}{w} = \overline{\underset{\sim}{v} \cdot \underset{\sim}{w}}$.

Ex. 1 Let $\underset{\sim}{v} = \begin{pmatrix} 1+i \\ 2-i \end{pmatrix}$. Then $\bar{\underset{\sim}{v}} = \begin{pmatrix} 1-i \\ 2+i \end{pmatrix}$. So

$$\|\underset{\sim}{v}\|^2 = \underset{\sim}{v}^T \bar{\underset{\sim}{v}} = \begin{pmatrix} 1+i & 2-i \end{pmatrix} \begin{pmatrix} 1-i \\ 2+i \end{pmatrix} = (1+i)(1-i) + (2-i)(2+i)$$

$$= (1-i^2)(2-i^2) = [1-(-1)] . [2-(-1)] = (2)(3) = 6.$$

$\therefore \|\underset{\sim}{v}\| = \sqrt{6}$.

Def. Let A be an $n \times n$ matrix. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if we can find a non-zero vector $\underset{\sim}{v} \in \mathbb{C}^n$ such that $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$. The non-zero vector $\underset{\sim}{v}$ is called an eigenvector of A belonging to the eigenvalue λ.

Qu: How can we find all the eigenvalues of A?

Sol. Suppose $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ & $\underset{\sim}{v} \neq 0$. Then $A\underset{\sim}{v} = \lambda I \underset{\sim}{v}$. So $(\lambda I - A)\underset{\sim}{v} = \underset{\sim}{0}$. Since $\underset{\sim}{v} \neq 0$, it follows that $(\lambda I - A)$ cannot be invertible. Hence $\det(\lambda I - A) = 0$.
So all possible eigenvalues can be found by solving the equation $\det(\lambda I - A) = 0$ for λ.

Ex. 2 Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all the eigenvalues of A and the corresponding eigenvectors.

Sol. Suppose $\det(\lambda I - A) = 0$. Then $\begin{vmatrix} \lambda - 0 & 1 \\ -1 & \lambda - 0 \end{vmatrix} = 0$

So $\lambda^2 - (-1) = 0$. $\therefore \lambda^2 + 1 = 0$. $\therefore \lambda = \pm i$

So the possible eigenvalues are $\lambda_1 = i$ & $\lambda_2 = -i$

(a) Suppose $\lambda_1 = i$ is really an eigenvalue. Then $(\lambda_1 I - A)\underset{\sim}{x} = \underset{\sim}{0}$ for some non-zero vector $\underset{\sim}{x}$. So

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \quad \begin{aligned} ix_1 + x_2 &= 0 \quad (1) \\ -x_1 + ix_2 &= 0 \quad (2) \end{aligned}$$

Multiplying (1) by i gives us

$$\begin{aligned} i^2 x_1 + i x_2 &= 0 \qquad \therefore -x_1 + ix_2 = 0 \qquad \therefore x_1 = ix_2 \\ -x_1 + i x_2 &= 0 \qquad\quad\;\; -x_1 + ix_2 = 0 \qquad\quad\; x_2 = \alpha \end{aligned}$$

So $\begin{pmatrix} i\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} i \\ 1 \end{pmatrix}$ will be an eigenvector if $\alpha \neq 0$.

Thus $\lambda_1 = i$ is really an eigenvalue of A.

(b) Suppose $\lambda_2 = -i$ is also an actual eigenvalue. Then $(\lambda_2 I - A)\underset{\sim}{x} = \underset{\sim}{0}$ for some non-zero vector $\underset{\sim}{x}$

So $$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \quad \begin{aligned} -ix_1 + x_2 &= 0 \quad (1) \\ -x_1 - ix_2 &= 0 \quad (2) \end{aligned}$$

Multiplying (1) by $-i$ gives us

$$\begin{aligned} i^2 x_1 - i x_2 &= 0 \qquad \therefore -x_1 - ix_2 = 0 \qquad \therefore x_1 = -ix_2 \\ -x_1 - i x_2 &= 0 \qquad\quad\;\; -x_1 - ix_2 = 0 \qquad\quad\; x_2 = \beta \end{aligned}$$

So $\begin{pmatrix} -i\beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} -i \\ 1 \end{pmatrix}$ will be an eigenvector if $\beta \neq 0$

Thus $\lambda_2 = -i$ is really an eigenvalue of A

§2 Properties of eigenvalues & eigenvectors

Prop. 1: Let A be an $n\times n$ matrix. Then A has n eigenvalues (counting multiplicities).

Proof: Suppose λ is an eigenvalue of A. Then $\det(\lambda I - A) = 0$. Now $\det(\lambda I - A)$ is a polynomial in λ of degree n. So $\det(\lambda I - A)$ will have n roots (counting multiplicities). Now each of these roots λ_i will indeed be an eigenvalue because if $\det(A - \lambda_i I) = 0$, then the system $(\lambda_i I - A)\underset{\sim}{x} = \underset{\sim}{0}$ will always have a non-trivial solution which will turn out to be an eigenvector corresponding to the eigenvalue λ_i. So A will have n eigenvalues (counting multiplicities).

Prop. 2: Suppose x_1 & x_2 are eigenvectors of A corresponding to different eigenvalues λ_1 and λ_2. Then the eigenvectors x_1 & x_2 are linearly independent.

Proof: Suppose $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$ are eigenvectors of A corresponding different eigenvalues λ_1

Then $A\underset{\sim}{x}_1 = \lambda_1 \underset{\sim}{x}_1$ and $A\underset{\sim}{x}_2 = \lambda_2 \underset{\sim}{x}_2$.

Now suppose $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2\}$ is linearly dependent.

Then $\underset{\sim}{x}_1 = c\,\underset{\sim}{x}_2$ for some $c \neq 0$ because $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \neq \underset{\sim}{0}$. So

$\lambda_1 \underset{\sim}{x}_1 = A\underset{\sim}{x}_1 = A(c\,\underset{\sim}{x}_2) = c(A\underset{\sim}{x}_2) = c(\lambda_2 \underset{\sim}{x}_2) = \lambda_2 (c\,\underset{\sim}{x}_2) = \lambda_2 \underset{\sim}{x}_1$

Hence $(\lambda_1 - \lambda_2)\underset{\sim}{x}_1 = \underset{\sim}{0}$. Since $\lambda_1 \neq \lambda_2$, we must have $\underset{\sim}{x}_1 = \underset{\sim}{0}$ which contradicts the fact that x_1 was an eigenvector of A. Hence x_1 & x_2 must be linearly independent.

Def. Let A be any $n \times n$ matrix. Then $\det(\lambda I - A)$ is a monic polynomial in λ of degree n in λ. It is called the <u>characteristic polynomial</u> $P_A(\lambda)$ of A. The equation $\det(\lambda I - A) = 0$ is called the <u>characteristic equation</u> of A.

Def. If λ is an eigenvalue of A, then we define the <u>eigenspace</u> $E_\lambda(A)$ by $E_\lambda(A) = \{\underset{\sim}{x} \in \mathbb{C}^n : A\underset{\sim}{x} = \lambda \underset{\sim}{x}\}$. Note that $\underset{\sim}{0} \in E_\lambda(A)$ and that all the non-zero vectors in $E_\lambda(A)$ will be eigenvectors corresponding to λ. The <u>geometric multiplicity</u> of λ is defined to be the dimension of $E_\lambda(A)$.

Def. Since $P_A(\lambda)$ is a polynomial of degree n, we can write $P_A(\lambda) = (\lambda - \lambda_1)^{n_1} \cdot (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$ where $\lambda_1, \dots, \lambda_k$ are the distinct roots of $P_A(\lambda)$. The multiplicity of the root λ_i is called the <u>algebraic multiplicity</u> of the eigenvalue λ. So the algebraic multiplicity of $\lambda_i = n_i$. Of course, $n_1 + n_2 + \cdots + n_k = n$.

<u>Theorem 3</u>: Let A be any $n \times n$ matrix and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A with the multiplicities of the roots taken into consideratio[n]
Then (a) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{Trace}(A)$, and
(b) $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A)$.

Proof (a) Since $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of A,

$$p_A(\lambda) = (\lambda - \lambda_1)\cdot(\lambda - \lambda_2) \cdot \ldots \cdot (\lambda - \lambda_n)$$
$$= \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n(\lambda_1 \cdots \lambda_n).$$

Now let us find $p_A(\lambda)$ by using Laplace's cofactor expansion of $\det(\lambda I - A)$. We have $p_A(\lambda) = \det(\lambda I - A)$

$$= (\lambda - a_{11})\cdot(-1)^{1+1}\det(M_{11}) + (-a_{12})\cdot(-1)^{1+2}\det(M_{12}) + \cdots + (-a_{1n})\cdot(-1)^{1+n}\det(M_{1n})$$

$$= (\lambda - a_{11}) \begin{vmatrix} \lambda - a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & \lambda - a_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & & \\ -a_{n2} & -a_{n3} & \cdots & \lambda - a_{nn} \end{vmatrix} + \underbrace{p_{n-2}(\lambda)}_{\text{polynomial of deg } (n-2) \text{ in } \lambda.}$$

$$= (\lambda - a_{11})\left[\lambda^{n-1} - (a_{22} + \cdots + a_{nn})\lambda^{n-2} + p_{n-3}(\lambda)\right] + p_{n-2}(\lambda)$$
$$= \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + q_{n-2}(\lambda)$$

So $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = Tr(A)$.

(b) Also if we put $\lambda = 0$, in the determinant, we get $p_A(0) = \det(0I - A) = \det(-A) = (-1)^n \det(A)$

But $p_A(\lambda) = (0 - \lambda_1)(0 - \lambda_2)\cdots(0 - \lambda_n) = (-1)^n\cdot(\lambda_1 \cdots \lambda_n)$.

So $\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$.

Prop. 4: (a) If A is a symmetric matrix, then all of its eigenvalues are real.

(b) AB & BA have the same set of eigenvalues.

Proof (a) Suppose A is symmetric. Then $A^T = A$. So if λ is an eigenvalue & $\underset{\sim}{x}$ is a corresponding eigenvector then $A^T \underset{\sim}{x} = \lambda \underset{\sim}{x}$ & $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$. Now $A\underset{\sim}{x} \cdot \underset{\sim}{x} = \underset{\sim}{x} \cdot (\overline{A\underset{\sim}{x}})$

So $(A\underset{\sim}{x})^T \bar{\underset{\sim}{x}} = \underset{\sim}{x}^T(\overline{A x})$. $\therefore \lambda \underset{\sim}{x}^T \bar{\underset{\sim}{x}} = \underset{\sim}{x}^T \bar{\lambda} \bar{\underset{\sim}{x}} = \bar{\lambda} \underset{\sim}{x}^T \bar{x}$. $\therefore \lambda = \bar{\lambda}$.

(b) Supp. $\underset{\sim}{x}$ is an eigenvector of AB corresp. to λ. Then $B\underset{\sim}{x} \neq \underset{\sim}{0}$ & $(BA)(B\underset{\sim}{x}) = B(AB)\underset{\sim}{x} = B\lambda \underset{\sim}{x} = \lambda(B\underset{\sim}{x})$. So λ is an eigenvalue of BA. The converse follows similarly.

§3. Diagonalization of matrices & matrix exponentiation

Theorem :: If an $n \times n$ matrix A has n linearly independent eigenvectors, then we can find an invertible matrix P such that $P^{-1}AP = D$ where D is the diagonal matrix consisting of the eigenvalues of A.

Ex. 1 Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

Sol. Suppose $\det(\lambda I - A) = 0$. Then $\begin{vmatrix} \lambda - 4 & 2 \\ -1 & \lambda - 1 \end{vmatrix} = 0$

So $(\lambda - 4)(\lambda - 1) - (-2) = 0$. $\therefore \lambda^2 - 5\lambda + 6 = 0$

$\therefore (\lambda - 3)(\lambda - 2) = 0$ $\therefore \lambda_1 = 3$ & $\lambda_2 = 2$.

So the eigen values are 3 & 2.

For $\lambda_1 = 3$, the system $(\lambda_1 I - A)\underset{\sim}{x} = \underset{\sim}{0}$ becomes

$$\begin{bmatrix} 3-4 & 2 \\ -1 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{So} \quad \begin{aligned} -x_1 + 2x_2 &= 0 \\ -x_1 + 2x_2 &= 0 \end{aligned}$$

$\therefore x_2 = \alpha$ and $x_1 = 2x_2 = 2\alpha$. So the eigenvectors corresponding to λ_1 are $\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with $\alpha \neq 0$.

For $\lambda_2 = 2$, the system $(\lambda_2 I - A)\underset{\sim}{x} = \underset{\sim}{0}$ becomes

$$\begin{bmatrix} 2-4 & 2 \\ -1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{So} \quad \begin{aligned} -2x_1 + 2x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

$\therefore x_2 = \beta$ and $x_1 = x_2 = \beta$. So the eigenvectors corresponding to λ_2 are $\beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $\beta \neq 0$.

Any set of 2 independent eigenvectors of A can be taken as the columns of the matrix P.

Ex.1 So choose $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ and

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D.$$

Ex.2 Let $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix

Sol. Suppose $\det(\lambda I - A) = 0$. Then $\begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda+1 \end{vmatrix} = 0$.

So $(\lambda-2)(\lambda+1) - 4 = 0$. $\therefore$ $\lambda^2 - \lambda - 6 = 0$.

$\therefore$ $(\lambda-3)(\lambda+2) = 0$. So $\lambda_1 = 3$ & $\lambda_2 = -2$ are the eigenvalues of A.

For $\lambda_1 = 3$, the system $(\lambda_1 I - A)\underset{\sim}{X} = \underset{\sim}{0}$ becomes

$$\begin{bmatrix} 3-2 & -2 \\ -2 & 3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \quad \begin{aligned} x_1 - 2x_2 &= 0 \\ -2x + 4x_2 &= 0 \end{aligned}$$

$\therefore$ $x_2 = \alpha$ and $x_1 = 2x_2 = 2\alpha$. So the eigenvectors corresponding to λ_1 are $\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with $\alpha \neq 0$.

For $\lambda_2 = -2$, the system $(\lambda_2 I - A)\underset{\sim}{X} = \underset{\sim}{0}$ becomes

$$\begin{bmatrix} -2-2 & -2 \\ -2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore \quad \begin{aligned} -4x_1 - 2x_2 &= 0 \\ -2x_1 - x_2 &= 0 \end{aligned}$$

$\therefore$ $x_2 = 2\beta$ $x_1 = -x_2/2 = -\beta$. So the eigenvectors corresponding to λ_2 are $\beta\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with $\beta \neq 0$.

If we take any two independent unit eigenvectors of A as the columns of P, P will be an orthogonal matrix.

Ex. 2 This is true because A was a <u>symmetric matrix</u>.

So take $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. Then $P^{-1} = P^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$.

So $P^{-1}AP = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$= \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 6/\sqrt{5} & 2/\sqrt{5} \\ 3/\sqrt{5} & -4/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

<u>Note</u>: If we take <u>any</u> two independent eigenvectors of A as the columns of a matrix Q, then we will still have $Q^{-1}AQ = D[\lambda_1, \lambda_2]$, but Q would not, in general, be orthogonal. For example, take $Q = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Then $Q^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

So $Q^{-1}AQ = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & -4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

<u>Ex. 3</u> Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Show that A has only one independent eigenvector and hence cannot be diagonalized.

<u>Sol.</u> Supp. $\det(\lambda I - A) = 0$. Then $\begin{vmatrix} \lambda-1 & -2 \\ 0 & \lambda-1 \end{vmatrix} = 0$. So $(\lambda-1)(\lambda-1) = 0$. $\therefore \lambda = 1$ (twice)

So $\lambda_1 = 1$ & $\lambda_2 = 1$. For $\lambda_1 = 1$, we have $\begin{bmatrix} 1-1 & -2 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $\left.\begin{matrix} 0x_1 - 2x_2 = 0 \\ 0x_1 - 0x_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1 = \alpha. \\ x_2 = 0 \end{matrix}$

$\therefore \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the only eigenvectors of A when $\alpha \neq 0$.

Since A does not have 2 indep. eigenvectors, A is not diagonalizable

Ex4 Let A be the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Diagonalize A.

Sol. $\det(\lambda I - A) = \begin{vmatrix} \lambda-0 & -1 \\ -1 & \lambda-0 \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow (\lambda-1)(\lambda+1) = 0.$

$\therefore \lambda_1 = 1$ and $\lambda_2 = -1$

For $\lambda_1 = 1$, we have $(\lambda_1 I - A)\underset{\sim}{X} = \underset{\sim}{0}$, so

$$\begin{bmatrix} 1-0 & -1 \\ -1 & 1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 1$, we have $(\lambda_2 I - A)\underset{\sim}{X} = \underset{\sim}{0}$, so

$$\begin{bmatrix} -1-0 & -1 \\ -1 & -1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{\sim}{0} \Rightarrow \underset{\sim}{X} = \beta\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, so $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

So the diagonal matrix that is similar to A is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. <u>Note</u>: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is also similar to A.

Example 5: Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues of A and one eigenvector for each eigenvalue. Then find the diagonal form that is similar to A.

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 + 1 \Rightarrow (\lambda - i)(\lambda + i) = 0$$
$$\therefore \lambda_1 = i \ \& \ \lambda_2 = -i$$

For $\lambda_1 = i$, we have $(A - \lambda_1 I)\underset{\sim}{x} = \underset{\sim}{0}$, so

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

For $\lambda_2 = -i$ we have $(A - \lambda_2 I)\underset{\sim}{x} = \underset{\sim}{0}$, so

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So we can take $\underset{\sim}{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ & $\underset{\sim}{u}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Le $P = [\underset{\sim}{u}_1, \underset{\sim}{u}_2] = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Then $P^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$

So

$$P^{-1}AP = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$
$$= \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \frac{1}{2i}\begin{bmatrix} 2i^2 & 0 \\ 0 & -2i^2 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The exponential of a matrix.

Def. Recall that $e^x = \sum_{k=0}^{\infty} (x^k/k!)$. Let A be an $n \times n$ matrix. We define e^A by

$$e^A = \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right) \cdot A^k. \qquad \text{(Here } A^0 = I_n \text{ for any } A.\text{)}$$

Fact: If $A = D[d_1, \ldots, d_n]$ is a diagonal matrix then $e^A = D[e^{d_1}, \ldots, e^{d_n}]$

Proof:
$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(D[d_1, \ldots, d_n]\right)^k$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} D[d_1^k, \ldots, d_n^k] =$$
$$= D\left[\sum_{k=0}^{\infty} \frac{1}{k!} d_1^k, \ldots, \sum_{k=0}^{\infty} \frac{1}{k!} d_n^k\right]$$
$$= D\left[e^{d_1}, \ldots, e^{d_n}\right].$$

Ex.: Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Find e^A.

Sol. We know that $P^{-1}AP = D$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. So $A = PDP^{-1}$. Thus

$$A^2 = (PDP^{-1})(PDP^{-1}) = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$
$$\vdots$$
$$A^k = (PDP^{-1})^k = PD^kP^{-1}$$

$$\therefore e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} (PDP^{-1})^k$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} PD^kP^{-1} = P\left(\sum_{k=1}^{\infty} \frac{1}{k!} D^k\right)P^{-1} = P(e^D)P^{-1}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^3 & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2e^3 & e^2 \\ e^3 & e^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2e^3 + e^2 & 2e^2 - 2e^3 \\ e^3 - e^2 & 2e^2 - e^3 \end{bmatrix} = \begin{bmatrix} e^2(2e+1) & 2e^2(1-e) \\ e^2(e-1) & e^2(2-e) \end{bmatrix}$$