MAS 3105 - LINEAR ALGEBRA FLORIDA INT'L UNIV.
TEST #1 - FALL 2020 TIME: 75 min.

Answer all 6 questions. No ***calculators, formula -sheets, or class-notes*** *are allowed. Show all working and justifications in problems 1-4. Provide all reasoning and justifications in problems 5 and 6. An unjustified answer will receive little or no credit.*

(20) 1. (a) Use *row operations* to transform the *augmented matrix* of the following system of linear equations into ***reduced row echelon form***.

(b) Then find the ***solution set*** of the system in standard form.

$$\begin{aligned} 2x_1 + 0x_2 - 4x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 - 3x_3 + 2x_4 &= -1 \\ 3x_1 - 2x_2 - 5x_3 + 6x_4 &= 5 \end{aligned}$$

(20) 2.(a) Define what it means for the nxn matrix A to be an ***invertible***.

(b) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -3 & 3 & 2 \\ -2 & 1 & 1 \end{bmatrix}$. Find A^{-1} by using ***row operations*** & verify that $AA^{-1} = I$.

(15) 3.(a) Let $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$. Find *det(B)* by using the ***Laplace cofactor expansion.***

(b) Check your answer in part (a) by using ***row operations.***

(15) 4.(a) Define what it means for the set, $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$, of vectors to be ***linearly independent***.

(b) If $\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix}$, and $\underline{v}_3 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$. Is $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ *linearly dependent?*

You must justify your answer by showing a relationship, if any, between them.

(15) 5.(a) Define what it means for a set, *S*, of vectors in $\mathbf{R}^n$ to be a ***subspace*** of $\mathbf{R}^n$; and what it means for the set, $B = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ of vectors to ***span*** *S*.

(b) If A, B, and C be mxn, nxp, and pxq matrices, resp. Prove that (AB)C = A(BC).

(15) 6. (a) Define what are the *(i,j)-**minor***, M_{ij}, and the *(i,j)-**cofactor***, A_{ij}, of an nxn matrix A.

(b) Let A be an nxn matrix and A^T be the ***transpose*** of the matrix A. Prove, by induction, that $\det(A^T) = \det(A)$, for each n > 0.

[*You may use the Laplace cofactor expansion of det(A), without proof, if needed.*]

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1(a) $$\left[\begin{array}{cccc|c} 2 & 0 & -4 & 4 & 2 \\ 1 & 2 & -3 & 2 & -1 \\ 3 & -2 & -5 & 6 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 1 \\ 0 & 2 & -1 & 0 & -2 \\ 0 & -2 & 1 & 0 & 2 \end{array}\right] \begin{array}{l} R1 := (1/2)R1 \\ R2 := R2 - (1/2)R1 \\ R3 := R3 - (3/2)R1 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & -1/2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R2 = (1/2)R2 \\ R3 := R3 + R2 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & -1/2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [A_R | b_R]$$

insert

(b) $$A_S = I_4 - A_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & -2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{\sim}{u} \quad \underset{\sim}{v}$$

$$\therefore \text{ Solution set} = \{b_R + \alpha u + \beta v\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

2(a) A is <u>invertible</u> if we can find an $n \times n$ matrix B such that $AB = I_n = BA$.

(b) $$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -3 & 3 & 2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2 := R2 + 3R1 \\ R3 := R3 + 2R1 \end{array}$$

$$\underbrace{}_{A} \quad \underbrace{}_{I_3}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -2 & 0 & -1 \end{array}\right] \begin{array}{l} R1 := R1 - R3 \\ R2 := R2 - R3 \\ R3 : (-1)R3 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & -1 \\ -3 & -1 & 0 \end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -3 & -1 & 0 \end{array}\right] \boxed{A^{-1}} \quad R3 := R3 - R2$$

<u>Check:</u> $$AA^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -3 & 3 & 2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & -1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1-1+3 & 0-1+1 & -1+1+0 \\ 3+3-6 & 0+3-2 & 3-3+0 \\ 2+1-3 & 0+1-1 & 2-1+0 \end{bmatrix} \checkmark$$

$$= I$$

3(a) $\det(B) = (-1)^{1+1}.2.\begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix} + (-1)^{1+2}.3.\begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} + (-1)^{1+3}.4.\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}$

$= 2(9+5) - 3(3+3) + 4(5-9) = 28 - 18 - 16 = \boxed{-6}$.

(b) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3/2 & -3 \\ 0 & -4 & 6 \end{bmatrix}$ $\begin{matrix} R2 := R2 - (1/2)R1 \\ R3 := R3 - (3/2)R1 \end{matrix}$

$\rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3/2 & -3 \\ 0 & 0 & -2 \end{bmatrix}$ $R3 := R3 + (8/3)R2$

$\therefore \det(B) = 2(3/2)(-2)$

$= \boxed{-6}$ again.

4(a) $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k\}$ is linearly independent if $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + \dots + c_k\underset{\sim}{v}_k = \underset{\sim}{0}$ implies that $c_1 = c_2 = \dots = c_k = 0$.

(b) Suppose $c_1\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then

$$\left.\begin{aligned} -c_1 - 3c_2 + 2c_3 &= 0 \\ c_1 + 3c_2 - 2c_3 &= 0 \\ 3c_1 + 4c_2 - c_3 &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} 0 &= 0 \\ c_1 + 3c_2 - 2c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned}\right\} \begin{aligned} &E1 := E1 + E2 \\ & \\ &E3 := E3 \dots \end{aligned}$$

$$\rightarrow \left.\begin{aligned} c_1 + c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned}\right\} E2 := E2 - 3E3$$

So c_3 is free, let $c_3 = \alpha$. Then $c_1 = -\alpha$, $c_2 = \alpha$.

Taking $\alpha = 1$ gives $(-1)\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + 1\begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, a concrete dependency relation between $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ & $\underset{\sim}{v}_3$. So $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ is linearly dependent. [You can also say that $\underset{\sim}{v}_1 = \underset{\sim}{v}_2 + \underset{\sim}{v}_3$ which shows that $\underset{\sim}{v}_1$ depends on $\underset{\sim}{v}_2$ & $\underset{\sim}{v}_3$.]

5(a) S is a <u>subspace</u> of $\mathbb{R}^n$ if (i) $S \neq \emptyset$, (ii) $\underset{\sim}{u}, \underset{\sim}{v} \in S \Rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in S$, and (iii) $\alpha \in \mathbb{R}$ and $\underset{\sim}{u} \in S \Rightarrow \alpha\underset{\sim}{u} \in S$.

$B = \{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$ spans S if every vector $\underset{\sim}{u} \in S$ can be written as $\underset{\sim}{u} = \alpha_1\underset{\sim}{v}_1 + \alpha_2\underset{\sim}{v}_2 + \dots \alpha_k\underset{\sim}{v}_k$ for some $\alpha_1, \dots, \alpha_k \in \mathbb{R}$.

5(b) $\{\underset{m\times n}{A}\underset{n\times p}{B})\underset{p\times q}{C}\}[i,j] = \sum_{l=1}^{p}(AB)[i,l].C[l,j]$

$$= \sum_{l=1}^{p}\left\{\sum_{k=1}^{n} A[i,k].B[k,l]\right\}.C[l,j] = \sum_{l=1}^{p}\sum_{k=1}^{n} A[i,k]B[k,l]C[l,j]$$

$$= \sum_{k=1}^{n} A[i,k].\left\{\sum_{l=1}^{p} B[k,l].C[l,j]\right\} = \sum_{k=1}^{n} A[i,k](BC)[k,j] = \{A(BC)\}[i,j]$$

$\therefore$ $(AB)C = A(BC)$ is always true.

6(a) M_{ij} is the $(n-1)\times(n-1)$ matrix that is obtained by deleting row i & column j of A. $A_{ij} = (-1)^{i+j}\det[M_{ij}]$

(b) We will prove $\det(A^T) = \det(A)$ by induction on n.

Basis: If $n=1$, then $\underset{1\times 1}{A} = [a_{11}]$, so $\det(A^T) = a_{11} = \det(A)$

Hence the result is true for $n=1$.

Ind. Step: Suppose the result is true for all $(n-1)\times(n-1)$ matrices. Then by expanding $\det(A^T)$ along the first column of A^T, we get for any $n\times n$ matrix:

$$\det(A^T) = \sum_{j=1}^{n} A^T[j,1].(-1)^{j+1}.\det([M_{j1}(A^T)])$$

$$= \sum_{j=1}^{n} A[1,j].(-1)^{1+j}.\det([M_{j1}(A)]^T)$$

$$= \sum_{j=1}^{n} A[1,j].(-1)^{1+j}.\det[M_{1j}(A)] = \det(A)$$

by expanding $\det(A)$ along the first row of A.

So if the result is true for $n-1$, it will also be true for n.

Hence by the Principle of Mathematical Induction, the result is true for all n, i.e., it is true for any $n\times n$ matrix A.

END.