MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #2 - FALL 2020 TIME: 75 min.

Answer all 6 questions. No notes or calculators are allowed. Show all working and provide all reasoning where required. An unjustified answer will receive little credit.

Let $E = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle$, $G = \langle \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\7\end{pmatrix} \rangle$, and $H = \langle \begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}3\\7\end{pmatrix} \rangle$.

(15) 1. (a) Find the *transition matrix*, ${}_E(I)_G$, from the basis G to the basis E; and the *transition matrix*, ${}_H(I)_E$, from E to H.

(b) Find the *transition matrix*, ${}_H(I)_G$, from the basis G to the basis H; and the *transition matrix*, ${}_G(I)_H$, from H to G.

Let $A = \begin{pmatrix} 1 & -2 & -1 & 1 \\ -2 & 4 & 10 & 2 \\ 2 & -4 & 0 & 3 \end{pmatrix}$.

(25) 2. (a) Find bases B & C for the *row space* and the *null space* of the matrix A.

(b) Find bases D and E for the *co-null space* and the *column space* of A.

(c) Verify your answers by checking that [B][C] = [O] and [D][E] = [O].

(15) 3. (a) Find the *best least squares fit* by *a linear function* $c_0 + c_1.x$ to the data below.

(b) *Check how close* your new answers are to the given values of $y(x)$.

x	-1	0	1	2
$y(x)$	-2	-1	0	2

x	-1	0	1	2
$c_0 + c_1.x$	?	?	?	?

(15) 4. Using the *Gram-Schmidt orthogonalization process*, find an *orthonormal basis* of the subspace $S = \text{span}\ \{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\4\\2\end{pmatrix}\}$ and *check* that it is orthonormal.

(15) 5.(a) Let V and W be vector spaces. Define what is a *linear transformation* from V to W and what it means for R to be a *subspace* of V.

(b) If L is a linear transformation from V to W and S is a subspace of W, prove that $L^{-1}[S]$ is a *subspace* of V.

(15) 6.(a) Define what is an *orthogonal set* of vectors in $\Re^n$ and what is the *orthogonal complement* $S^\perp$ of a subset S of W. (Here $\Re$ = set of real numbers)

(b) Suppose that $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is an orthogonal set of vectors in $\Re^n$ with $\| \underline{v}_j \| = j$ for each $j = 1, 2, 3, \ldots, k$. Prove that $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is *linearly independent*.

MAS 3105 – Linear Algebra   Florida Internat. Univ.

Solutions to Test #2   Fall 2020

1 (a) $_E(I)_G = \begin{bmatrix}1&2\\3&7\end{bmatrix}$, $\quad {_H(I)_E} = \{_E(I)_H\}^{-1} = \begin{bmatrix}2&3\\5&7\end{bmatrix}^{-1} = \frac{1}{14-15}\begin{bmatrix}7&-3\\-5&2\end{bmatrix}$

$$= \begin{bmatrix}-7&3\\5&-2\end{bmatrix}$$

(b) $_H(I)_G = {_H(I)_E}\,{_E(I)_G} = \begin{bmatrix}-7&3\\5&-2\end{bmatrix}\begin{bmatrix}1&2\\3&7\end{bmatrix} = \begin{bmatrix}2&7\\-1&-4\end{bmatrix}$

$_G(I)_H = \{_H(I)_G\}^{-1} = \frac{1}{-8+7}\begin{bmatrix}-4&-7\\1&2\end{bmatrix} = \begin{bmatrix}4&7\\-1&-2\end{bmatrix}$

Check $\quad {_G(I)_H} = {_G(I)_E}\,{_E(I)_H} = \frac{1}{1}\begin{bmatrix}7&-2\\-3&1\end{bmatrix}\begin{bmatrix}2&3\\5&7\end{bmatrix} = \begin{bmatrix}4&7\\-1&-2\end{bmatrix}$ ✓

2 (a) $\begin{bmatrix}1&-2&-1&1\\-2&4&10&2\\2&-4&0&3\end{bmatrix} \rightarrow \begin{bmatrix}1&-2&-1&1\\0&0&8&4\\0&0&2&1\end{bmatrix}\begin{matrix}\\R2 \leftarrow R2+2R1\\R3 \leftarrow R3-2R1\end{matrix}$

$\rightarrow \underbrace{\begin{bmatrix}1&-2&0&3/2\\0&0&1&1/2\\0&0&0&0\end{bmatrix}}_{A_{RR}}\begin{matrix}R1 \leftarrow R1+\frac{1}{8}R2\\R2 \leftarrow \frac{1}{8}R2\\R3 \leftarrow R3-\frac{1}{4}R2\end{matrix} \rightarrow \underbrace{\begin{bmatrix}1&-2&0&3/2\\0&0&0&0\\0&0&1&1/2\\0&0&0&0\end{bmatrix}}_{A_S}, \quad I_4 - A_S = \begin{bmatrix}0&2&0&-3/2\\0&1&0&0\\0&0&0&-1/2\\0&0&0&1\end{bmatrix}$

Basis of Row space of A = non-zero rows of A_S

$\therefore \quad B = \{(1,-2,0,3/2),\ (0,0,1,1/2)\}$

Basis of null-space of A = non-zero columns of $(I_4 - S)$

$\therefore \quad C = \{(2,1,0,0)^T,\ (-3/2,0,-1/2,1)^T\}$

(c) $\underset{2\times4}{[B]}\ \underset{4\times2}{[C]} = \underset{2\times4}{\begin{bmatrix}1&-2&0&3/2\\0&0&1&1/2\end{bmatrix}}\ \underset{4\times2}{\begin{bmatrix}2&-3/2\\1&0\\0&-1/2\\0&1\end{bmatrix}} = \underset{2\times2}{\begin{bmatrix}0&0\\0&0\end{bmatrix}}$

Check

(b) $\left[\begin{array}{cccc|ccc}1&-2&-1&1&1&0&0\\-2&4&10&2&0&1&0\\2&-4&0&3&0&0&1\end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc}1&-2&-1&1&1&0&0\\0&0&8&4&2&1&0\\0&0&2&1&-2&0&1\end{array}\right]\begin{matrix}\\R2 \leftarrow R2+2R1\\R3 \leftarrow R3-2R1\end{matrix}$

$\rightarrow \left[\begin{array}{cccc|ccc}1&-2&-1&1&1&0&0\\0&0&8&4&2&1&0\\0&0&8&4&-8&0&4\end{array}\right] R3 \leftarrow 4R3 \longrightarrow \begin{array}{c}\left[\begin{array}{cccc|ccc}1&-2&-1&1&1&0&0\\0&0&8&4&2&1&0\\0&0&0&0&-10&-1&4\end{array}\right]\\ \uparrow 1st \qquad \uparrow 3rd \qquad \underbrace{\qquad\qquad}_{A_{CN}}\end{array} R3 \leftarrow R3-R2$

A_R

2(b) Basis of CoNullspace of A = rows of A_{CN} with zeros rows in the left-side matrix

$\therefore D = \{(-10, -1, 4)\}$

A Basis of ColSpace of = columns of A corresp. to the leading elements of A_R

$\therefore E = \{(1, -2, 2)^T, (-1, 10, 0)^T\}$

(c) Check

$$\underset{1\times3}{[D]}\,\underset{3\times2}{[E]} = \underset{1\times3}{[-10, -1, 4]} \underset{3\times2}{\begin{bmatrix} 1 & -1 \\ -2 & 10 \\ 2 & 0 \end{bmatrix}} = \underset{1\times2}{\begin{bmatrix} 0 & 0 \end{bmatrix}}$$

3(a) Suppose $y(x) = c_0 + c_1 x$. Then

$$c_0 + c_1(-1) = -2$$
$$c_0 + c_1(0) = -1$$
$$c_0 + c_1(1) = 0$$
$$c_0 + c_1(2) = 2$$

$$\therefore \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix}}_{b} \qquad \text{So} \quad A^T A \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = A^T b$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \quad \therefore \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \frac{1}{24-4} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -18/20 \\ 26/20 \end{bmatrix} = \begin{bmatrix} -9/10 \\ 13/10 \end{bmatrix}$$

Check

x	-1	0	1	2
$\hat{c}_0 + \hat{c}_1 x$	$\frac{-22}{10}$	$\frac{-9}{10}$	$\frac{4}{10}$	$\frac{17}{10}$

close enough! ✓

4(a) $\underset{\sim}{v}_1 = x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\quad v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{\|v_1\|^2} \right) v_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$

$$\underset{\sim}{v}_3 = \underset{\sim}{x}_3 - \left(\frac{x_3 \cdot v_1}{\|v_1\|^2} \right) v_1 - \left(\frac{x_3 \cdot v_2}{\|v_2\|^2} \right) v_2 = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{12}{12} \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Check $\quad v_1 \cdot v_2 = (-2) + (0) + 2 = 0$, $\; v_1 \cdot v_3 = 1 + 0 - 1 = 0$, $\; v_2 \cdot v_3 = -2 + 4 - 2 = 0$

Orthonormal set $\quad \hat{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\; \hat{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\; \hat{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

5(a) A linear transformation is any function $L : V \to W$ such that for any $\underline{u}, \underline{v} \in V$ and $\alpha \in \mathbb{R}$ (i) $L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v})$ and (ii) $L(\alpha u) = \alpha L(u)$. $\cdot$ R is a subspace of V if $R \neq \emptyset$ and for any $u, v \in R$ & $\alpha \in \mathbb{R}$, we have $u + v \in R$ & $\alpha u \in R$.

5(b) $L^{-1}[S] = \{x \in V : L(x) \in S\}$ Now $L(0_V) = L(0.0_V) = 0.L(0_V) = 0_W$
So $L(0_V) = 0_W \in S$ because 0_W is in any subspace of W.
So $0_V \in L^{-1}[S]$ and hence $L^{-1}[S] \neq \emptyset$. Now suppose that $x_1, x_2 \in L^{-1}[S]$ & $\alpha \in \mathbb{R}$. Then $L(x_1) \in S$ & $L(x_2) \in S'$
So $L(x_1 + x_2) = L(x_1) + L(x_2) \in S'$ bec. S is a subspace
$\therefore$ $x_1 + x_2 \in S'$. Also $L(\alpha x_1) = \alpha L(x_1) \in S'$ because S is a subspace. So $\alpha x_1 \in L^{-1}(S')$. Hence
$x_1, x_2 \in L^{-1}[S]$ & $\alpha \in \mathbb{R} \Rightarrow x_1 + x_2 \in L^{-1}[S']$ & $\alpha x_1 \in L^{-1}[S']$
Thus $L^{-1}[S]$ is a subspace of V.

6(a) A set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ is orthogonal if for any two distinct vectors $u, v \in \mathcal{S}$, $u.v = 0$. The orthogonal complement $S^{\perp}$ of a subspace S of $\mathbb{R}^n$ is defined by $S^{\perp} = \{x \in \mathbb{R}^n : x.u = 0 \text{ for all } u \in S\}$.

6(b) Let $\{v_1, v_2, \ldots, v_k\}$ be a set of orthogonal vectors in $\mathbb{R}^n$ with $\|v_j\| = j$. Then $v_i . v_j = 0$ for each $1 \leq i, j \leq k$ with $i \neq j$. Now suppose $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v = 0$. Then for each j, $(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_j v_j + \cdots + \alpha_k v_k) . v_j = 0 . v_j$
So $\alpha_1 (v_1 . v_j) + \alpha_2 (v_2 . v_j) + \cdots \alpha_j (v_j . v_j) + \cdots + \alpha_k . (v_k . v_j) = 0$
$\therefore$ $0 + 0 + \cdots + \alpha_j (j)^2 + \cdots + 0 = 0$
$\therefore$ $\alpha_j = 0$ because $j^2 \neq 0$. Since this is true for each j, it follows that $\alpha_1 = \alpha_2 = \cdots = \alpha_j = \cdots = \alpha_k = 0$
$\therefore$ $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$
$\therefore$ $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set of vectors.