MAS 3105 - LINEAR ALGEBRA — FLORIDA INT'L UNIV.

TEST #1 - FALL 2021 — TIME: 75 min.

Answer all 6 questions. No ***calculators, formula-sheets, or class-notes*** *are allowed. Show all working and justifications in problems 1-4. Provide all reasoning and justifications in problems 5 and 6. An unjustified answer will receive little or no credit.*

(20) 1. Use *row operations* to transform the *augmented matrix* of the **following system** of linear equations into ***reduced row echelon form***. Then find the ***solution set*** of the system in ***standard form***.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 &= 0 \\ 2x_1 + 4x_2 - 4x_3 + 0x_4 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 &= -2 \end{aligned}$$

(20) 2.(a) Define what it means for the nxn matrix A to be ***invertible***.

(b) Let $A = \begin{bmatrix} 1 & -3 & -1 \\ -1 & 4 & 1 \\ 0 & -2 & -1 \end{bmatrix}$. Find A^{-1} by using ***row operations*** & verify that $AA^{-1} = I$.

(15) 3.(a) Let $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 3 & 5 & 3 \end{bmatrix}$. Find *det(B)* by using ***row operations***.

(b) Check your answer in part (a) by using the ***Laplace cofactor expansion***.

(15) 4. (a) Define what it means for the set, $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$, of vectors to be ***linearly dependent***.

(b) Let $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}$, and $\underline{v}_3 = \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix}$. Is $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ *linearly dependent?*

You must justify your answer by showing a relationship, if any, between them.

(15) 5. (a) Define what it means for a set, *S,* of vectors in $\mathbf{R}^n$ to be a ***subspace*** of $\mathbf{R}^n$; and what it means for the set of vectors, $B = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ to ***span*** *S.*

(b) Let A be any mxn matrix, B be any mxn matrix, and C be any nxp matrix. Prove that we always have $(B - A)C = (BC) - (AC)$.

(15) 6. (a) Define what are the ***(i,j)-minor***, M_{ij}, and the ***(i,j)-cofactor***, A_{ij}, of an nxn matrix A.

(b) Let A be an nxn matrix and A' be the matrix obtained by interchanging *row i* and *row j* of A. Prove, by induction, that $\det(A') = -\det(A)$, for each $n \geq 2$. [*You may use the Laplace cofactor expansion of det(A), without proof, if needed.*]

MAS 3105 - Linear Algebra   Florida Int'l Univ.
Solutions to Test #1   Fall 2021.

1. (a)

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ 2 & 4 & -4 & 0 & 2 \\ -1 & -2 & 1 & 2 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 2 \\ 0 & 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} \text{pivot row} \\ R2 := R2 - 2R1 \\ R3 := R3 + R1 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & 0 & -4 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} R1 := R1 + \frac{3}{2}R2 \\ R2 := \frac{1}{2}R2 \\ R3 := R3 + R2 \end{array} \to \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{1 \quad 2 \quad 0 \quad -4}}^{A_s}} & b_s \\ 1 & 2 & 0 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) $I_4 - A_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} & u & & v \end{array} \\ \begin{bmatrix} 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

Sol. set $= \{ b_s + \alpha u + \beta v : \alpha, \beta \in \mathbb{R} \} = \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

2 (a) A is invertible if we can find an $n \times n$ matrix B s.t. $AB = I_n = BA$.

(b)

$$\left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \quad R2 := R2 + R1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 4 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array}\right] \begin{array}{l} R1 := R1 + 3R2 \\ \\ R3 := R3 + 2R1 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix} \qquad \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right] \begin{array}{l} R1 := R1 - R3 \\ \\ R3 := (-1)R3 \end{array}$$

Check: $AA^{-1} = \begin{bmatrix} 1 & -3 & -1 \\ -1 & 4 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

3(a) $\det(B) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 3 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -3 & 6 \\ 0 & -4 & 6 \end{vmatrix}$ $R2 := R2 - 2R1$, $R3 := R3 - 3R1$

$= \begin{vmatrix} 1 & 3 & -1 \\ 0 & -3 & 6 \\ 0 & 0 & -2 \end{vmatrix} = (1)(-3)(-2) = \boxed{6}$ $R3 := R3 - \frac{4}{3}R2$

(b) $\det(B) = 1.(-1)^{1+1}\begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix} + 3.(-1)^{3+1}.\begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix}$ ✓

$= 1.(9-20) + (-2)(9-(-5)) + 3.(12-(-3)) = -11 - 28 + 45 = \boxed{6}$

4(a) $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_k\}$ is linearly dependent if we can find scalars $c_1, c_2, \cdots, c_k$ at least one of which is nonzero s.t. $c_1 \underset{\sim}{v}_1 + \cdots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}$

(b) Suppose $c_1\begin{pmatrix}1\\-1\\3\end{pmatrix} + c_2\begin{pmatrix}-4\\4\\2\end{pmatrix} + c_3\begin{pmatrix}-3\\3\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Then

$\left.\begin{aligned} c_1 - 4c_2 - 3c_3 &= 0 \\ -c_1 + 4c_2 + 3c_3 &= 0 \\ -3c_1 + 2c_2 + 4c_3 &= 0 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} c_1 - 4c_2 - 3c_3 &= 0 \\ 0 &= 0 \\ -10c_2 - 5c_3 &= 0 \end{aligned}\right\}$ $E2 := E2 + E1$, $E3 := E3 + 3E1$

$\therefore 5c_3 = -10c_2$ & so $c_3 = -2c_2$. Also $c_1 = 3c_3 + 4c_2 = -6c_2 + 4c_2 = -2c_2$

Taking $c_2 = -1$, gives $c_1 = 2$ & $c_3 = 2$. So $2\underset{\sim}{v}_1 - \underset{\sim}{v}_2 + 2\underset{\sim}{v}_3 = \underset{\sim}{0}$

<u>Check</u> : $2\begin{pmatrix}1\\-1\\3\end{pmatrix} - \begin{pmatrix}-4\\4\\2\end{pmatrix} + 2\begin{pmatrix}-3\\3\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓ $\therefore \{v_1, v_2, v_3\}$ is linearly dependent.

5(a) S is a subspace of $\mathbb{R}^n$ if S is <u>non-empty</u>, $\underset{\sim}{u}, \underset{\sim}{v} \in S \Rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in S$, and $\alpha \in \mathbb{R}$ & $\underset{\sim}{u} \in S \Rightarrow \alpha \underset{\sim}{u} \in S$. $B = \{\underset{\sim}{v}_1, \cdots, \underset{\sim}{v}_k\}$ spans S if every vector $\underset{\sim}{u} \in S$ can be written in the form $\underset{\sim}{u} = \alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \cdots + \alpha_k \underset{\sim}{v}_k$ with $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$.

(b) $\{\underset{m\times n}{(B-A)}\underset{n\times p}{C}\}[i,j] = \sum_{k=1}^{n}\{(B-A)[i,k].C[k,j]\}$

$= \sum_{k=1}^{n}\{(B[i,k] - A[i,k]).C[k,j]\}$

$= \sum_{k=1}^{n}\{(B[i,k].C[k,j]) - (A[i,k].C[k,j])\}$

$= \sum_{k=1}^{n}\{B[i,k].C[k,j]\} - \sum_{k=1}^{n}\{A[i,k].C[k,j]\}$

$= (BC)[i,j] - (AC)[i,j] = \{(BC) - (AC)\}[i,j]$. $\therefore (B-A)C = (BC) - (AC)$

6(a) The (i,j)-minor $M_{i,j}(A)$ is the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j of A. The (i,j)-cofactor of A is defined by $A_{ij} = (-1)^{i+j}\det[M_{i,j}(A)]$.

(b) We shall prove that $\det(A') = -\det(A)$ by induction on n. For $n=2$, we have only two rows. So

$$\det(A') = \det\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} = a_{21}a_{12} - a_{11}a_{22}$$
$$= -(a_{11}a_{22} - a_{21}a_{12})$$
$$= \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det(A)$$

So the result is true for $n=2$.

Suppose the result is true for all $(n-1)\times(n-1)$ matrices, $n \geq 3$. Let A' = the matrix obtained from A by switching row i & row j. We will expand $\det(A')$ along row k for some $k \neq i$ or j. This choice of k is possible because $n \geq 3$. Then

$$\det(A') = \sum_{l=1}^{n} (A')[k,l] \cdot (A')_{k,l}$$

$$= \sum_{l=1}^{n} A[k,l] \cdot (-1)^{k+l} \det[M_{k,l}(A')]$$ because row k of A' = row k of A.

$$= \sum_{l=1}^{n} A[k,l] \cdot (-1)^{k+l} \cdot (-1) \cdot \det[M_{k,l}(A)]$$

because $M_{k,l}(A') = M_{k,l}(A)$ with two rows interchanged.

$$= -\sum_{l=1}^{n} \underbrace{A[k,l] \cdot (-1)^{k+l} \cdot \det[M_{k,l}(A)]}$$

$$= -\det(A)$$ since this is the expansion of $\det A$ along row k.

So if the result is true for $n-1$, it will be true for n. Hence by the Principle of Mathematical Induction, the result is true for all n, i.e., for all $n\times n$ matrices with $n \geq 2$.

END