MAS 3305 - LINEAR ALGEBRA | FLORIDA INT'L UNIV.
TEST #2 - FALL 2021 | TIME: 75 min.

Answer all 6 questions. No notes or calculators are allowed. Show all working and provide all reasoning where required. An unjustified answer will receive little credit.

Let $E=\left\langle\binom{1}{0},\binom{0}{1}\right\rangle$, $G=\left\langle\binom{5}{2},\binom{3}{1}\right\rangle$, and $H=\left\langle\binom{2}{1},\binom{3}{2}\right\rangle$.

(15) 1. (a) Find the *transition matrix*, ${}_G(I)_E$, from the *basis E to the basis G*; and the *transition matrix*, ${}_H(I)_E$, from the *basis E to the basis H*.

(b) Find the *transition matrix*, ${}_H(I)_G$, from the basis G to the basis H; and the *transition matrix*, ${}_G(I)_H$, from the basis H to the basis G.

Let $A = \begin{pmatrix} 1 & -2 & -1 & -1 \\ -1 & 2 & 5 & 3 \\ 2 & -4 & 0 & -1 \end{pmatrix}$.

(25) 2. (a) Find bases B & C for the *row space* and the *null space* of the matrix A.

(b) Find bases D and F for the *co-null space* and the *column space* of A.

(c) Verify your answers by checking that [B][C] = [O] and [D][F] = [O].

(15) 3.(a) Let $\underline{v}$ & $\underline{w}$ be non-zero vectors in $\Re^n$. Define $proj_{\underline{v}}(\underline{w})$ and $orthog_{\underline{v}}(\underline{w})$.

(b) Find the shortest distance between the line $\Lambda = \{\alpha\,(2, 1, -1)^T : \alpha \in \Re\}$ and the point $P = (1, 2, -2)^T$.

(15) 4. (a) Find the *best least squares fit* by *a linear function* $c_0 + c_1.x$ to the data below.

(b) Check *how close* your *best fit* answers are to the given values of $y(x)$.

x	−1	0	1	2
$y(x)$	3	2	0	−2

x	−1	0	1	2
$c_0 + c_1.x$	?	?	?	?

(15) 5. (a) Using the *Gram-Schmidt orthogonalization process*, find an *orthogonal basis* of the subspace, $S = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$ and *check* that it is orthogonal.

(b) Using part (a), find *an orthonormal basis* of the subspace S.

(15) 6.(a) Define what is an *orthogonal set* of vectors in $\Re^n$ and what is the *orthogonal complement* $S^\perp$ of a subspace S of $\Re^n$. (Here $\Re$ = set of real numbers).

(b) If L is a *linear transformation* from V to W and R is a subspace of V, prove that L[R] is also a *subspace* of W.

MAS 3105 - Linear Algebra Florida International Univ.

Solutions to Test #2 Fall 2021 (1)

1 (a) $_G(I)_E = \{_E(I)_G\}^{-1} = \begin{bmatrix}5 & 3\\ 2 & 1\end{bmatrix}^{-1} = \frac{1}{5-6}\begin{bmatrix}1 & -3\\ -2 & 5\end{bmatrix} = \begin{bmatrix}-1 & 3\\ 2 & -5\end{bmatrix}$

$_H(I)_E = \{_E(I)_H\}^{-1} = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}^{-1} = \frac{1}{4-3}\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix} = \begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$

(b) $_H(I)_G = {}_H(I)_E\,{}_E(I)_G = \begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}\begin{bmatrix}5 & 3\\ 2 & 1\end{bmatrix} = \begin{bmatrix}4 & 3\\ -1 & -1\end{bmatrix}$

$_G(I)_H = {}_G(I)_E\,{}_E(I)_H = \begin{bmatrix}-1 & 3\\ 2 & -5\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 3\\ -1 & -4\end{bmatrix}$

<u>Check</u>: $_G(I)_H = \{_H(I)_G\}^{-1} = \begin{bmatrix}4 & 3\\ -1 & -1\end{bmatrix}^{-1} = \frac{1}{(-4)+3}\begin{bmatrix}-1 & -3\\ 1 & 4\end{bmatrix} = \begin{bmatrix}1 & 3\\ -1 & -4\end{bmatrix}$ ✓

2(a) $\begin{bmatrix}1 & -2 & -1 & -1\\ -1 & 2 & 5 & 3\\ 2 & 4 & 0 & -1\end{bmatrix} \to \begin{bmatrix}1 & -2 & -1 & -1\\ 0 & 0 & 4 & 2\\ 0 & 0 & 2 & 1\end{bmatrix}$ $R2 \leftarrow R2 + R1$, $R3 \leftarrow R3 - 2R1$

$\to \begin{bmatrix}1 & -2 & 0 & -1/2\\ 0 & 0 & 1 & 1/2\\ 0 & 0 & 0 & 0\end{bmatrix}$ $R1 \leftarrow R1 + (1/4)R2$, $R2 \leftarrow (1/4)R2$, $R3 \leftarrow R3 - (1/2)R2$ $A_s = \begin{bmatrix}1 & -2 & 0 & -1/2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1/2\\ 0 & 0 & 0 & 0\end{bmatrix}$

$I_4 - A_s = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}1 & -2 & 0 & -1/2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1/2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 2 & 0 & 1/2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1/2\\ 0 & 0 & 0 & 1\end{bmatrix}$

$\therefore$ Basis of RowSp(A) $= \{(1, -2, 0, -1/2), (0, 0, 1, 1/2)\} = [B]$

and basis of NullSp(A) $= \{(2, 1, 0, 0)^T, (1/2, 0, -1/2, 1)^T\} = [C]$

$[B][C] = \begin{bmatrix}1 & -2 & 0 & -1/2\\ 0 & 0 & 1 & 1/2\end{bmatrix}\begin{bmatrix}2 & 1/2\\ 1 & 0\\ 0 & -1/2\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ ✓

(b) $[A|I_3] = \left[\begin{array}{cccc|ccc}1 & -2 & -1 & -1 & 1 & 0 & 0\\ -1 & 2 & 5 & 3 & 0 & 1 & 0\\ 2 & 4 & 0 & -1 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{cccc|ccc}1 & -2 & -1 & -1 & 1 & 0 & 0\\ 0 & 0 & 4 & 2 & 1 & 1 & 0\\ 0 & 0 & 2 & 1 & -2 & 0 & 1\end{array}\right]$ $R2 \leftarrow R2 + R1$, $R3 \leftarrow R3 - 2R1$

$\to \left[\begin{array}{cccc|ccc}1 & -2 & -1 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1/2 & 1/4 & 1/4 & 0\\ 0 & 0 & 0 & 0 & -5/2 & -1/2 & 1\end{array}\right]$ $R2 \leftarrow (1/4)R2$, $R3 \leftarrow R3 - (1/2)R2$

$D = \{(-5/2, -1/2, 1)\}$

$F = \left\{\begin{pmatrix}1\\ -1\\ 2\end{pmatrix}, \begin{pmatrix}-1\\ 5\\ 0\end{pmatrix}\right\}$

Check $[D][F] = \begin{bmatrix}-\frac{5}{2}, & -\frac{1}{2}, & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 5\\ 2 & 0\end{bmatrix} = [0\ 0]$ ✓

3(a) The projection of w onto v is defined by $\text{proj}_v(w) = \left\{\frac{(w\cdot v)}{\|v\|^2}\right\} v$ and $\text{orthog}_v(w) = w - \left(\frac{w\cdot v}{\|v\|^2}\right) v$. ②

(b) Let Q be the point on $\Lambda = \{\alpha(2,1,-1)^T : \alpha \in \mathbb{R}\}$ that is closest to P. Then $Q = (2\alpha_0, \alpha_0, -\alpha_0)^T$ & we will have $\vec{PQ} \perp (2,1,-1)^T$

Now $\vec{PQ} = \vec{PO} + \vec{OQ} = -\vec{OP} + \vec{OQ} = -\begin{pmatrix}1\\2\\-2\end{pmatrix} + \begin{pmatrix}2\alpha_0\\ \alpha_0\\ -\alpha_0\end{pmatrix} = \begin{pmatrix}2\alpha_0 - 1\\ \alpha_0 - 2\\ -\alpha_0 + 2\end{pmatrix}$. So

$0 = \vec{PQ}\cdot(2,1,-1)^T = \begin{pmatrix}2\alpha_0 - 1\\ \alpha_0 - 2\\ -\alpha_0 + 2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix} \Rightarrow (4\alpha_0 - 2) + (\alpha_0 - 2) + (\alpha_0 - 2) = 0$

$\therefore 6\alpha_0 - 6 = 0 \Rightarrow \alpha_0 = 1.$

$\therefore \vec{PQ} = \begin{pmatrix}2-1\\1-2\\-1+2\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$. So shortest distance from P to Λ will be $\|\vec{PQ}\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$.

4(a) Suppose $y(x) = c_0 + c_1 x$. Then

$c_0 + c_1(-1) = 3$
$c_0 + c_1(0) = 2$
$c_0 + c_1(1) = 0$
$c_0 + c_1(2) = -2$.

$\therefore \underbrace{\begin{bmatrix}1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}}_{A}\begin{bmatrix}c_0\\c_1\end{bmatrix} = \underbrace{\begin{bmatrix}3\\2\\0\\-2\end{bmatrix}}_{b}$. So $A^TA\begin{bmatrix}\hat{c}_0\\\hat{c}_1\end{bmatrix} = A^T b$ (best-fit equation

$\therefore \begin{bmatrix}1 & 1 & 1 & 1\\-1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\hat{c}_0\\\hat{c}_1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\-1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}3\\2\\0\\-2\end{bmatrix}.$

$\therefore \begin{bmatrix}4 & 2\\2 & 6\end{bmatrix}\begin{bmatrix}\hat{c}_0\\\hat{c}_1\end{bmatrix} = \begin{bmatrix}3\\-7\end{bmatrix}$, So $\begin{bmatrix}\hat{c}_0\\\hat{c}_1\end{bmatrix} = \frac{1}{24-4}\begin{bmatrix}6 & -2\\-2 & 4\end{bmatrix}\begin{bmatrix}3\\-7\end{bmatrix} = \frac{1}{20}\begin{bmatrix}32\\-34\end{bmatrix} = \begin{bmatrix}16/10\\-17/10\end{bmatrix}$

$\therefore y = \frac{16}{10} - \frac{17x}{10}$. (b) <u>Check</u>:

x	-1	0	1	2
$\hat{c}_0 + \hat{c}_1 x$	$\frac{-33}{10}$	$\frac{16}{10}$	$\frac{1}{10}$	$\frac{18}{10}$

5(a) Let $v_1 = x_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$. Then $v_2 = x_2 - \left(\frac{x_2\cdot v_1}{\|v_1\|^2}\right)v_1 = \begin{pmatrix}2\\2\\-1\end{pmatrix} - \frac{3}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$

and $v_3 = x_3 - \left(\frac{x_3\cdot v_1}{\|v_1\|^2}\right)v_1 - \left(\frac{x_3\cdot v_2}{\|v_2\|^2}\right)v_2 = \begin{pmatrix}0\\1\\2\end{pmatrix} - \frac{3}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{-3}{6}\begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

<u>Check</u>: $v_1\cdot v_2 = 1(1) + 1(1) + 1(-2) = 0$ $\quad v_1\cdot v_3 = 1(-1/2) + 1(1/2) + 1(0) = 0$

$v_2\cdot v_3 = 1(-1/2) + 1(1/2) + (-2)(0) = 0$ So $\{v_1, v_2, v_3\}$ is our orthogonal basis.

(b) $u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$, $u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{1/2}}\begin{pmatrix}-1/2\\1/2\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$.

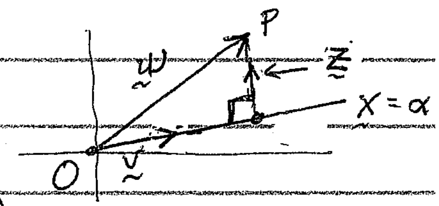
Our orthonormal basis will be $\{u_1, u_2, u_3\}$.

6(a). Let $\mathcal{S}$ be a set of vectors in $\mathbb{R}^n$. We say that $\mathcal{S}$ is an orthogonal set if for any two distinct vectors $\underset{\sim}{u}$ & $\underset{\sim}{v}$ in $\mathcal{S}$ we always have $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$. The orthogonal complement of a subspace S of $\mathbb{R}^n$ is defined by $S^{\perp} = \{\underset{\sim}{y} \in \mathbb{R}^n : \underset{\sim}{x} \cdot \underset{\sim}{y}$ for each $\underset{\sim}{x} \in S\}$

(b) We are given that R is a subspace of V and that $L: V \to W$ is a linear transformation. To show that $L[R]$ is a subspace of W we must show that (i) $L[R] \neq \emptyset$, (ii) $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2 \in L[R] \Rightarrow \underset{\sim}{Y}_1 + \underset{\sim}{Y}_2 \in L[R]$, and (iii) $\underset{\sim}{Y} \in R$ & $\alpha \in \mathbb{R} \Rightarrow \alpha \underset{\sim}{Y} \in L[R]$.

Now $R \neq \emptyset$ because R is a subspace of V. So we can find an $\underset{\sim}{x}_0 \in R$. $\therefore$ $L(x_0) \in L[R]$ and so $L[R] \neq \emptyset$
Suppose $Y_1, Y_2 \in L[R]$. Then we can find $x_1, x_2 \in R$ such that $\underset{\sim}{Y}_1 = L(x_1)$ & $\underset{\sim}{Y}_2 = L(\underset{\sim}{x}_2)$ by the definition of $L[R]$.
So $\underset{\sim}{Y}_1 + \underset{\sim}{Y}_2 = L(x_1) + L(x_2) = L(x_1 + x_2) \in L[R]$ bec. L is a linear transformation. Finally suppose $\alpha \in \mathbb{R}$ and $Y \in L[R]$. Then we can find an $x \in R$ such that $L(\underset{\sim}{x}) = \underset{\sim}{Y}$. So $\alpha \underset{\sim}{y} = \alpha L(x) = L(\alpha \underset{\sim}{x}) \in L[R]$ because L is linear. Hence $L[R]$ is a subspace of W. END

3(b) (Alternative solution): Let $\underset{\sim}{w} = \vec{OP} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ & $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. Put $\underset{\sim}{z} = \text{orthog}_{\underset{\sim}{v}}(\underset{\sim}{w})$. Then shortest distance from P to the line $x = \alpha \underset{\sim}{v}$ will be $\|\underset{\sim}{w}\|$. Now



$$\underset{\sim}{z} = \underset{\sim}{w} - \left(\frac{\underset{\sim}{w} \cdot \underset{\sim}{v}}{\|\underset{\sim}{v}\|^2} \right) \underset{\sim}{v}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \left(\frac{2+2+2}{(\sqrt{4+1+1})^2} \right) \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$\therefore$ shortest distance $= \|\underset{\sim}{z}\| = \sqrt{(-1)^2 + (1)^2 + (-1)^2} = \sqrt{3}$.