MAS 3305 - LINEAR ALGEBRA　　　　FLORIDA INT'L UNIV.

TEST #1 - SPRING 2013　　　　TIME: 75 min.

Answer all 6 questions. NO CALCULATORS or CELL PHONES ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(15) 1. Find the ***solution set*** of the following system of equations by using ***Gaussian elimination and back substitution*** on the equations.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -1 \\ 2x_1 + x_2 + 2x_3 &= 3 \\ -x_1 + 3x_2 - 2x_3 &= 4 \end{aligned}$$

(20) 2. Use row operations to transform the ***augmented matrix*** of the following system into ***reduced row echelon form***. Then find the ***solution set*** of the system.

$$\begin{aligned} x_1 + x_2 + 0x_3 - 2x_4 &= 1 \\ x_1 + 2x_2 - x_3 - 5x_4 &= -3 \\ -x_1 - 2x_2 + 2x_3 + 8x_4 &= 5 \end{aligned}$$

(20) 3. Let $A = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 1 & -1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 1 \\ 1 & -2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) Find A^{-1} by using ***row operations*** & check that $AA^{-1} = I$.
(b) Find det(B) by using the ***cofactor expansion***.

(15) 4(a) Define what it means for the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ to be ***linearly independent***.

(b) **Use your definition** to determine whether or not the three vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ are linearly independent.

(15) 5. Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ be vectors from a vector space V.

(a) Define what is span$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$.
(b) Prove that span$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is a subspace of V.

(15) 6(a) Define what is a ***subspace*** of a vector space V.

(b) Let A, B & C be an m×n, n×p, and p×q matrices respectively. Prove that (AB)C = A(BC). [*Show each step of the proof.*]

MAS 3105 - LINEAR ALGEBRA FLORIDA INTERNATIONAL UNIV.

SOLUTIONS TO TEST # 1 SPRING 2013

1.
$$\left.\begin{array}{l} x_1 - 2x_2 + x_3 = -1 \\ 2x_1 + x_2 + 2x_3 = 3 \\ -x_1 + 3x_2 - 2x_3 = 4 \end{array}\right\} \rightarrow \left.\begin{array}{l} x_1 - 2x_2 + x_3 = -1 \\ 5x_2 + 0x_3 = 5 \\ x_2 - x_3 = 3 \end{array}\right\} \begin{array}{l} \\ E2 := E2 - 2E1 \\ E3 := E3 + E1 \end{array}$$

$$\rightarrow \left.\begin{array}{l} x_1 - 2x_2 + x_3 = -1 \\ x_2 = 1 \\ -x_3 = 2 \end{array}\right\} \begin{array}{l} \\ E2 := (1/5)E2 \\ E3 := E3 - (1/5)E2 \end{array}$$

$\therefore$ $x_3 = -2$, $x_2 = 1$, and $x_1 = 2x_2 - x_3 - 1 = 2(1) - (-2) - 1 = 3$

$\therefore$ Solution set $= \left\{ \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \right\}$

2.
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 1 \\ 1 & 2 & -1 & -5 & -3 \\ -1 & -2 & 2 & 8 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & -3 & -4 \\ 0 & -1 & 2 & 6 & 6 \end{array}\right] \begin{array}{l} \\ R2 := R2 - R1 \\ R3 := R3 + R1 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & -1 & -3 & -4 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right] \begin{array}{l} R1 := R1 - R2 \\ \\ R3 := R3 + R2 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right] \begin{array}{l} R1 := R1 - R3 \\ R2 := R2 + R3 \\ \\ \end{array}$$

x_4 is free. Put $x_4 = t$. Then $x_3 = 2 - 3x_4 = 2 - 3t$, $x_2 = -2$, and $x_1 = 3 + 2x_4 = 3 + 2t$. Hence

Solution set $= \left\{ \begin{pmatrix} 3+2t \\ -2 \\ 2-3t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$.

3(a)
$$\left[\begin{array}{ccc|ccc} 1 & -2 & -5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & -4 & 0 & 0 & 1 \end{array}\right] \quad (A \mid I) \to \left[\begin{array}{ccc|ccc} 1 & -2 & -5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \quad R3 := R3 - R1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R1 := R1 + 2R2 \\ \\ R3 := R3 - R2 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R1 := R1 - R3 \\ R2 := R2 + 2R3 \\ R3 := (-1)R3 \end{array}$$

$$(I \mid A^{-1})$$

Check: $AA^{-1} = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

(b) Expanding along the third row gives us:

$$\det(B) = 1.(-1)^{3+1} \cdot \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} + 0.(-1)^{3+2} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 2.(-1)^{3+3} \cdot \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix}$$

$$= 1.(12 - (-2)) + 0 + 2.(-4 - 4) = 14 - 16 = -2.$$

4(a) The vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_n}$ are linearly independent if $c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \cdots + c_n \underset{\sim}{v_n} = \underset{\sim}{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$.

(b) Suppose $c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\left.\begin{array}{l} c_1 - 2c_2 + 3c_3 = 0 \\ 0c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 + 2c_3 = 0 \end{array}\right\} \to \left.\begin{array}{r} c_1 - 2c_2 + 3c_3 = 0 \\ c_2 + c_3 = 0 \\ 3c_2 - c_3 = 0 \end{array}\right\} \quad E3 := E3 - E1$$

$$\to \left.\begin{array}{r} c_1 - 2c_2 + 3c_3 = 0 \\ c_2 + c_3 = 0 \\ -4c_3 = 0 \end{array}\right\} \quad E3 := E3 - 3E2$$

$\therefore -4c_3 = 0 \Rightarrow c_3 = 0$. So $c_2 = -c_3 = 0$, and $c_1 = 2c_2 - 3c_3 = 0$.
Hence the 3 vectors are linearly independent.

5(a) $\text{span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_k\} = \{\alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \cdots + \alpha_k \underset{\sim}{v}_k : \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$

(You can also say that $\text{span}\{v_1, v_2, \ldots, v_k\}$ is the set of all linear combinations of $v_1, v_2, \ldots, v_k$.)

(b) Let $S = \text{span}\{v_1, v_2, \ldots, v_k\}$. Then S is non-empty because $0.v_1 + 0.v_2 + \cdots + 0.v_k = 0 \in S$. Now suppose $x, y \in S$. Then $x = \alpha_1 v_1 + \cdots + \alpha_k v_k$ & $y = \beta_1 v_1 + \cdots + \beta_k v_k$ for some scalars $\alpha_i, \beta_i \in \mathbb{R}$. So

$$x + y = (\alpha_1 v_1 + \cdots + \alpha_k v_k) + (\beta_1 v_1 + \cdots + \beta_k v) = (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_k + \beta_k) v_k \in S$$

Also if α is any scalar, then

$$\alpha x = \alpha(\alpha_1 v_1 + \cdots + \alpha_k v_k) = (\alpha\alpha_1) v_1 + \cdots + (\alpha\alpha_k) v_k \in S.$$

Hence S is a subspace of V.

6(a) A subspace of the vector space V is any non-empty subset S of vectors from V such that

(i) for any $x, y \in S$ we have $x + y \in S$,

(ii) for any $x \in S$ and $\alpha \in \mathbb{R}$ we have $\alpha x \in S$.

(b)
$$\{(AB)C\}[i,j] = \sum_{k=1}^{p} \{(AB)[i,k] . C[k,j]\}$$

$$= \sum_{k=1}^{p} \left\{ \left[\sum_{l=1}^{n} A[i,l] . B[l,k] \right] . C[k,j] \right\}$$

$$= \sum_{k=1}^{p} \left\{ \sum_{l=1}^{n} [A[i,l] . B[l,k] . C[k,j]] \right\}$$

$$= \sum_{l=1}^{n} \left\{ \sum_{k=1}^{p} [A[i,l] . B[l,k] . C[k,j]] \right\}$$

$$= \sum_{l=1}^{n} \left\{ A[i,l] . \sum_{k=1}^{p} [B[l,k] . C[k,j]] \right\}$$

$$= \sum_{l=1}^{n} \{ A[i,l] . (BC)[l,j] \} = \{A(BC)\}[i,j]$$

$$\therefore \quad (AB)C = A(BC).$$