MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #2 - SPRING 2013 TIME: 75 min.

Answer all 6 questions. NO CALCULATORS or CELL PHONES ARE ALLOWED. Show all working and provide all reasoning where required. An unjustified answer will receive little or no credit.

Let E = the standard basis of $\mathbf{R}^2$, G = $[\underline{u}_1, \underline{u}_2]$, and H = $[\underline{v}_1, \underline{v}_2]$ where $\underline{u}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix}3\\5\end{bmatrix}$ and $\underline{v}_1 = \begin{bmatrix}-1\\2\end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix}-1\\3\end{bmatrix}$.

(15) 1(a) Find the **transition matrix**, ${}_E(I)_G$, from the basis G to the basis E; and the **transition matrix**, ${}_H(I)_E$, from E to H.

(b) Find the **transition matrix**, ${}_H(I)_G$, from the basis G to the basis H; and the **transition matrix**, ${}_G(I)_H$, from H to G.

Also let $A = \begin{bmatrix}1 & 2 & -1 & 3\\1 & 2 & 0 & 1\\-1 & -2 & -1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & -2 & 2\\-2 & 4 & -4\\1 & -2 & 3\\-1 & 2 & 0\end{bmatrix}$.

(20) 2(a) Find bases for the **row space** & **null space** of the matrix A.

(b) If C is the matrix with the basis of *RowSp(A)* as its rows and D is the matrix with the basis of *Null(A)* as its columns, **check** that CD = **O**.

(20) 3(a) Find bases for the **co-null space** & **column space** of the matrix B.

(b) If F is the matrix with the basis of *CoNull(B)* as its rows and K is the matrix with the basis of *ColSp(B)* as its columns, **check** that FK = **O**.

(15) 4. Find the **best least squares fit** by a **linear function** $c_0 + c_1.x$ to the data below and then **check how close** your answers are to y(x).

x	0	1	2
y(x)	-2	0	1

x	0	1	2
$c_0 + c_1.x$	?	?	?

(15) 5(a) Define what is an ***orthonormal basis*** of $\mathbf{R}^n$.

(b) Using the **Gram-Schmidt process**, find an orthonormal basis C from the basis B = $[\underline{u}_1, \underline{u}_2, \underline{u}_3]$ where $\underline{u}_1 = \begin{bmatrix}0\\0\\3\end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix}4\\3\\2\end{bmatrix}$, $\underline{u}_3 = \begin{bmatrix}1\\2\\4\end{bmatrix}$.

(15) 6(a) Define what is a ***linear transformation*** from the vector space V to the vector space W.

(b) If L is a linear transformation from V to W and S is a subspace of V, prove that L[S] is a **subspace** of W.

MAS 3105 - Linear Algebra — Florida International Univ.

Solutions to Test #2 — Spring 2013.

1(a)(i) $_E(I)_G = [\underset{\sim}{u}_1\ \underset{\sim}{u}_2] = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ ✓ $\quad {_G}(I)_E = [_E(I)_G]^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

(ii) $_H(I)_E = [_E(I)_H]^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{-3+2}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$ ✓

(b)(i) $_H(I)_G = {_H}(I)_E\ {_E}(I)_G = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -5 & -14 \\ 4 & 11 \end{bmatrix}$ ✓

(ii) $_G(I)_H = {_G}(I)_E\ {_E}(I)_H = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$ ✓

Check: $_G(I)_H = [_H(I)_G]^{-1} = \begin{bmatrix} -5 & -14 \\ 4 & 11 \end{bmatrix}^{-1} = \frac{1}{56-55}\begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$.

2(a) $\underbrace{\begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 2 & 0 & 1 \\ -1 & -2 & -1 & 1 \end{bmatrix}}_{=A} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$ $\quad R2 := R2 - R1$, $\quad R3 := R3 + R1$

$\sim \begin{bmatrix} \underline{1} & 2 & 0 & 1 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad R1 := R1 + R2$, $\quad R3 = R3 + 2R2$

reduced row echelon form $= A_R =$ the last matrix.

(i) A basis of row space $(A) = \{(1, 2, 0, 1), (0, 0, 1, -2)\}$ ✓

(ii) To find a basis of Null(A) we can solve the system $A_R \underset{\sim}{x} = \underset{\sim}{0}$

$\therefore \left.\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_3 - 2x_4 &= 0 \end{aligned}\right\} \Rightarrow x_4 = t,\ x_3 = 2t$
$x_2 = s,\ x_1 = -2s - t$.

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s-t \\ s \\ 2t \\ t \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}; \quad s, t \in \mathbb{R}$.

$\therefore$ a basis of Null$(A) = \left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}\right\}$. ✓

2(b) $BC = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

3(a)
$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 1 & -2 & 3 \\ -1 & 2 & 0 \end{matrix}}_{=B} & \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I_4} \end{array}\right] \sim \left[\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2 := R2+2R1 \\ R3 := R3-R1 \\ R4 := R4+R1 \end{array}$$

row echelon form B_E =
$$\sim \left[\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ R2 := R3 \\ R3 := R2 \\ R4 := R4-2R3 \end{array}$$

(i) A basis of co-null$(B) = \{(2,1,0,0), (3,0,-2,1)\}$ ✓

(ii) Since the pivot entries of the row echelon form of B_E of B are in the 1st & 3rd columns of B, a basis of ColSp(B) will consist of the 1st & 3rd columns of B, i.e., $\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 3 \\ 0 \end{pmatrix}\right\}$ ✓

3(b) So $FK = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

4(i) If we assume $y(x) = c_0 + c_1 x$, this leads us to the 3 equations on the right.
$$\begin{cases} c_0 + c_1 \cdot 0 = -2 \\ c_0 + c_1 \cdot 1 = 0 \\ c_0 + c_1 \cdot 2 = 1 \end{cases}$$
But the equations are inconsistent, so we have to find the best least squares solution to the system, $\hat{c}_0 + \hat{c}_1 x$

$\therefore \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\underset{\sim}{b}}$. Normal Eq. is $A^T A \underset{\sim}{\hat{c}} = A^T \underset{\sim}{b}$.

So $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. So $\begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -11 \\ 9 \end{bmatrix} = \begin{bmatrix} -11/6 \\ 3/2 \end{bmatrix}$.

$\therefore \hat{c}_0 = -11/6, \; \hat{c}_1 = 3/2$. ✓

(ii) Check:

x	0	1	2
$\hat{c}_0 + \hat{c}_1 x$	$-11/6$	$1/3$	$7/6$ ✓

This is close enough to:

x	0	1	2
y	-2	0	1

5(a) $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_n\}$ is an orthormal basis of $\mathbb{R}^n$ if $\|\underset{\sim}{v}_i\| = 1$ for each $i = 1, \dots, n$ and $\underset{\sim}{v}_i \cdot v_j = 0$ for all $i \neq j$.

(b) $$\underset{\sim}{v}_1 = \frac{\underset{\sim}{u}_1}{\|\underset{\sim}{u}_1\|} = \frac{1}{\sqrt{9+0+0}}\begin{pmatrix}0\\0\\3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \checkmark$$

$$\underset{\sim}{p}_2 = (\underset{\sim}{u}_2 \cdot \underset{\sim}{v}_1)\underset{\sim}{v}_1 = \left[\begin{pmatrix}4\\3\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\begin{pmatrix}0\\0\\1\end{pmatrix} = 2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$v_2 = \frac{[\underset{\sim}{u}_2 - p_2]}{\|\underset{\sim}{u}_2 - p_2\|} = \frac{1}{\|\underset{\sim}{u}_2 - \underset{\sim}{p}_2\|}\left[\begin{pmatrix}4\\3\\2\end{pmatrix} - \begin{pmatrix}0\\0\\2\end{pmatrix}\right] = \frac{1}{\sqrt{4^2+3^2+0^2}}\begin{pmatrix}4\\3\\0\end{pmatrix} = \begin{pmatrix}4/5\\3/5\\0\end{pmatrix} \checkmark.$$

$$\underset{\sim}{p}_3 = [\underset{\sim}{u}_3 \cdot \underset{\sim}{v}_1]\underset{\sim}{v}_1 + [\underset{\sim}{u}_3 \cdot \underset{\sim}{v}_2]\underset{\sim}{v}_2 = \left[\begin{pmatrix}1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\begin{bmatrix}0\\0\\1\end{bmatrix} + \left[\begin{pmatrix}1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}4/5\\3/5\\0\end{pmatrix}\right]\begin{pmatrix}4/5\\3/5\\0\end{pmatrix}$$

$$= 4\begin{pmatrix}0\\0\\1\end{pmatrix} + 2\begin{pmatrix}4/5\\3/5\\0\end{pmatrix} = \begin{pmatrix}8/5\\6/5\\4\end{pmatrix}$$

$$\underset{\sim}{v}_3 = \frac{[\underset{\sim}{u}_3 - \underset{\sim}{p}_3]}{\|\underset{\sim}{u}_3 - \underset{\sim}{p}_3\|} = \frac{1}{[\underset{\sim}{u}_3 - \underset{\sim}{p}_3]}\left[\begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}8/5\\6/5\\4\end{pmatrix}\right] = \frac{1}{\sqrt{\frac{9}{25}+\frac{16}{25}+0}}\begin{pmatrix}-3/5\\4/5\\0\end{pmatrix} = \begin{pmatrix}-3/5\\4/5\\0\end{pmatrix} \checkmark.$$

$\therefore$ the orthonormal basis $= \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}4/5\\3/5\\0\end{pmatrix}, \begin{pmatrix}-3/5\\4/5\\0\end{pmatrix}\right\}$.

6(a) A linear transformation from V to W is a function $L: V \to W$ such that $L(\underset{\sim}{x}+\underset{\sim}{y}) = L(\underset{\sim}{x}) + L(\underset{\sim}{y})$ & $L(\alpha \underset{\sim}{x}) = \alpha L(\underset{\sim}{x})$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in V$ & $\alpha \in \mathbb{R}$.

(b) We know that $L[S] = \{L(\underset{\sim}{x}) : \underset{\sim}{x} \in S\}$. Since S is a subspace of V, $S \neq \emptyset$, so we can find a vector $\underset{\sim}{x}_0 \in S$. Then $L(\underset{\sim}{x}_0) \in L[S]$, so $L[S] \neq \emptyset$ ✓

Now suppose that $\underset{\sim}{w}_1$ and $\underset{\sim}{w}_2$ are in $L[S]$. Then we can find $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in S$ such that $\underset{\sim}{w}_1 = L(\underset{\sim}{x}_1)$ and $\underset{\sim}{w}_2 = L(\underset{\sim}{x}_2)$. Since S is a subspace, $\underset{\sim}{x}_1 + \underset{\sim}{x}_2 \in S$.

So $\underset{\sim}{w}_1 + \underset{\sim}{w}_2 = L(\underset{\sim}{x}_1) + L(\underset{\sim}{x}_2) = L(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) \in L[S]$ ✓

Finally suppose $\underset{\sim}{w} \in L[S]$ and $\alpha \in \mathbb{R}$. Then we can an $\underset{\sim}{x}$ in S such that $\underset{\sim}{w} = L(\underset{\sim}{x})$. Since S is a subspace, $\alpha \underset{\sim}{x} \in S$. So $\alpha \underset{\sim}{w} = \alpha L(\underset{\sim}{x}) = L(\alpha \underset{\sim}{x}) \in L[S]$ ✓

Hence $L[S]$ is a subspace of W.

ALTERNATIVE SOLUTIONS (not for the faint of heart)

2(a)(ii) We found the reduced row echelon form A_R of A on the first page of the solutions. We can get a basis of Null(A) by inserting rows of zeros between the non-zero rows of A_R to get a square matrix A_s with the leading 1's in the diagonal position, as follows.

$$A_s = \begin{bmatrix} \underline{1} & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I} - \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_s} = \begin{array}{c} \begin{bmatrix} 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \quad\uparrow \text{ basis } \uparrow \end{array}$$

The non-zero columns of $I - A_s$ will give us a basis of $n(A)$ which is the same as the one found by solving $A_R\underset{\sim}{x} = \underset{\sim}{0}$.

3(b)(i) We can find a basis of Co-null(B) by finding a basis of B^T and then transposing the column vectors in your basis to get row vectors.

$$B^T = \begin{bmatrix} 1 & -2 & 1 & -1 \\ -2 & 4 & -2 & 2 \\ 2 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \\ R2 := R2 + 2R1 \\ R3 := R3 - 2R1 \end{array}$$

$$\sim (B^T)_R = \begin{bmatrix} \underline{1} & -2 & 0 & -3 \\ 0 & 0 & \underline{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R1 := R1 - R3 \\ R2 := R3 \\ R3 := R2 \end{array}$$

$$(B^T)_S = \begin{bmatrix} \underline{1} & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \quad\uparrow \text{ basis } \uparrow \end{array}$$

So a basis of Co-Null(B) = $\{(2,1,0,0), (3,0,-2,1)\}$.

$\{(1,-2,0,-3)^T, (0,0,1,2)^T\}$ is also another basis for ColSp(B).