MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #1 - FALL '03 TIME: 75 min.

Answer all 6 questions. NO CALCULATORS ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(15) 1. Find the solution set of the following system of equations by using Gaussian elimination and back substitution.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 - x_3 &= -8 \\ -x_1 + 2x_2 + x_3 &= -1 \end{aligned}$$

(20) 2. Use row operations to transform the augmented matrix of the following system into reduced row echelon form. Then find the solution set of the system.

$$\begin{aligned} x_1 + 2x_2 - x_3 - 3x_4 &= 1 \\ x_1 + 3x_2 - x_3 - 5x_4 &= 2 \\ x_1 - 2x_2 + 2x_3 + 5x_4 &= 6 \end{aligned}$$

(15) 3. (a) Use the cofactor expansion to find $\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -1 & -4 & 1 \end{vmatrix}$.

(b) Check your answer by using row operations.

(15) 4. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$. Find A^{-1} and check that $AA^{-1}=I$.

(15) 5. (a) Define what are the (i,j) minor and the (i,j) cofactor of a matrix A.

(b) Let A be an n×n matrix and B be the matrix obtained by multiplying row i of A by α. Prove that
$$\det(B) = \alpha . \det(A).$$

(20) 6. (a) Define what is a subspace of a vector space V.

(b) Let A be an m×n matrix and B be an n×p matrix. Prove that
$$(AB)^T = B^TA^T.$$ (Show each step of the proof.)

MAS 3105 - Linear Algebra — Florida Int'l Univ.

Solutions to Test #1 — Fall 2003

1. We have
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 - x_3 &= -8 \\ -x_1 + 2x_2 + x_3 &= -1 \end{aligned}$$

$\therefore$
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 & & \\ -x_2 - 3x_3 &= -10 & & E2 := E2 - 2E1 \\ 4x_2 + 2x_3 &= 0 & & E3 := E3 + E1 \end{aligned}$$

$\therefore$
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 & & \\ -x_2 - 3x_3 &= -10 & & \\ -10x_3 &= -40 & & E3 := E3 + 4E2 \end{aligned}$$

$\therefore$ $-10x_3 = -40 \Rightarrow x_3 = 4$

$-x_2 - 3(4) = -10 \Rightarrow x_2 = 10 - 12 = -2$

$x_1 + 2(-2) + 4 = 1 \Rightarrow x_1 = 1 + 4 - 4 = 1$

$\therefore$ Solution set $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right\}$

2. $$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -3 & 1 \\ 1 & 3 & -1 & -5 & 2 \\ 1 & -2 & 2 & 5 & 6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & -3 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & -4 & 3 & 8 & 5 \end{array}\right] \begin{array}{l} \\ R2 := R2 - R1 \\ R3 := R3 - R1 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 3 & 0 & 9 \end{array}\right] \begin{array}{l} R1 := R1 - 2R2 \\ \\ R3 := R3 + 4R2 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 9 \end{array}\right] \begin{array}{l} \\ \\ R3 := \frac{1}{3}R3 \end{array}$$

2.

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \quad R1 := R1 + R3$$

$$\therefore \begin{array}{llll} x_1 & & & + x_4 = 2 \\ & x_2 & & -2x_4 = 1 \\ & & x_3 & = 3 \end{array}$$

$\therefore$ $x_4 = t$

$x_3 = 3$

$x_2 = 1 + 2x_4 = 1 + 2t$

$x_1 = 2 - x_4 = 2 - t$

$$\text{Sol. set} = \left\{ \begin{bmatrix} 2-t \\ 1+2t \\ 3 \\ t \end{bmatrix} ;\ t \in \mathbb{R} \right\}.$$

3(a)

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -1 & -4 & 1 \end{vmatrix} = 1.(-1)^{1+1}.\begin{vmatrix} -1 & -1 \\ -4 & 1 \end{vmatrix} + (-2).(-1)^{1+2}.\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + 1.(-1)^{1+3}.\begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix}$$

$$= 1.(-1-4) + 2(2-1) + 1.(-8-1)$$

$$= -5 + 2 - 9 = -14 + 2 = -12$$

(b)

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -1 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & -6 & 2 \end{vmatrix} \quad \begin{array}{l} R2 := R2 - 2R1 \\ R3 := R3 + R1 \end{array}$$

$$= \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & -4 \end{vmatrix} \quad R3 := R3 + 2R2$$

$$= (1)(3)(-4) = -12.$$

4(a)

$$\left[\underbrace{\begin{array}{ccc} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 3 & -1 \end{array}}_{A} \middle| \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R2 := R2 + 2R1 \\ R3 := R3 - 2R1 \end{array}$$

4(a)

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R1 := R1 - R2 \\ \\ R3 := R3 - R2 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array}\right] \quad R1 := R1 + R3$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$

Check: $AA^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} -5 & 2 & 1 \\ 2 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

5(a) The (i,j) minor of A is the submatrix M_{ij} of A obtained by deleting row i & column j from A.
The (i,j) cofactor of A is given $A_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$

(b) $\det(B) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}$ expand along row i

$$= (\alpha a_{i1}) A_{i1} + (\alpha a_{i2}) A_{i2} + \cdots + (\alpha a_{in}) A_{in}$$
$$= \alpha [a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}] = \alpha \det(A).$$

6(a) A subspace of V is any subset S of V such that
(i) $S \neq \emptyset$, (ii) $\underline{x}, \underline{y} \in S \Rightarrow \underline{x} + \underline{y} \in S$, and (iii) $\alpha \in \mathbb{R}$ & $\underline{x} \in S \Rightarrow \alpha \underline{x} \in S$.

(b) We have

$$\begin{aligned} (AB)^T[i,j] &= (AB)[j,i] && \text{by def. of transpose} \\ &= \sum_{k=1}^{n} A[j,k] \cdot B[k,i] && \text{by def of matrix mult.} \\ &= \sum_{k=1}^{n} B[k,i] \cdot A[j,k] && \text{commutativity of "}\cdot\text{"} \\ &= \sum_{k=1}^{n} (B^T)[i,k] \cdot (A^T)[k,j] && \text{def of transpose} \\ &= (B^T)(A^T)[i,j] && \text{by def of matrix mult.} \end{aligned}$$

$\therefore (AB)^T = B^T A^T$.