MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #2 - FALL '03 TIME: 75 min.

Answer all 6 questions. NO CALCULATORS ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(15) 1. (a) Define what it means for the vectors $\underline{v}_1, \ldots, \underline{v}_n$ to span R^k.

(b) Use your definition to determine whether or not the following three vectors span R^3: $\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(15) 2. (a) Define what it means for the vectors $\underline{v}_1, \ldots, \underline{v}_n$ to be linearly independent.

(b) Use your definition to determine whether or not the three vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ are linearly independent.

(15) 3. Find a basis for the column space and a basis for the null space of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ -1 & 1 & -1 & -1 \\ 2 & -2 & 1 & 4 \end{bmatrix}.$$

(20) 4. (a) Find the point in the plane $x-2y+2z = -2$ that is closest to the point $(2,2,3)^T$.

(b) Find the shortest distance between the point $(3,2,0)^T$ and the line $\underline{x} = (1,0,-4)^T + \alpha(2,1,2)^T$.

(15) 5. Let L be a linear transformation from the vector space V to the vector space W. Prove that

(a) $L(\underline{0}_V) = \underline{0}_W$ and

(b) $\ker(L)$ is a subspace of V

(20) 6. (a) Define what is an orthogonal set of vectors in R^n

(b) If $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is an orthogonal set of unit vectors in R^n prove that $\{\underline{v}_1, \ldots, \underline{v}_k\}$ must be linearly independent.

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Solutions to Test # 2 | Fall 2003

1(a) $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n$ span $\mathbb{R}^k$ if every vector $\underset{\sim}{v} \in \mathbb{R}^k$ can be expressed in the form $\underset{\sim}{v} = \alpha_1 \underset{\sim}{v}_1 + \alpha \underset{\sim}{v}_2 + \cdots + \alpha_n \underset{\sim}{v}_n$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

(b) We will try to see if an arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ can be expressed as a linear combination of the 3 given vectors. Suppose $\alpha_1 \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then

$$\left.\begin{aligned} \alpha_1 \qquad + \alpha_3 &= a \\ -\alpha_1 + \alpha_2 + 2\alpha_3 &= b \\ -3\alpha_1 + \alpha_2 \qquad &= c \end{aligned}\right\} \longrightarrow \left.\begin{aligned} \alpha_1 \qquad + \alpha_3 &= a \\ \alpha_2 + 3\alpha_3 &= b+a \\ \alpha_2 + 3\alpha_3 &= c+3a \end{aligned}\right\} \begin{aligned} & \\ & E2 := E2 + E1 \\ & E3 := E3 + 3E1 \end{aligned}$$

$$\longrightarrow \left.\begin{aligned} \alpha_1 \qquad + \alpha_3 &= a \\ \alpha_2 + 3\alpha_3 &= b+a \\ 0 &= c+2a-b \end{aligned}\right\} \begin{aligned} & \\ & \\ & E3 := E3 - E2 \end{aligned}$$

So if we take $a=0$, $b=0$, and $c=1$, the system will be inconsistent. Hence $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ cannot be expressed as a linear combination of the 3 given vectors. So the three vectors do not span $\mathbb{R}^3$.

2(a) The vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ are linearly independent if the equation $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_n \underset{\sim}{v}_n = \underset{\sim}{0}$ implies that $c_1 = c_2 = \cdots = c_n = 0$.

(b) Suppose $c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \underset{\sim}{0}$. Then

$$\left.\begin{aligned} c_1 \qquad + c_3 &= 0 \\ -c_1 - 2c_2 - c_3 &= 0 \\ c_2 + 2c_3 &= 0 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} c_1 \qquad + c_3 &= 0 \\ -2c_2 \qquad &= 0 \\ c_2 + 2c_3 &= 0 \end{aligned}\right\} \begin{aligned} & \\ & E2 := E2 + E1 \\ & \end{aligned}$$

$$\longrightarrow \left.\begin{aligned} c_1 \qquad + c_3 &= 0 \\ c_2 \qquad &= 0 \\ 2c_3 &= 0 \end{aligned}\right\} \begin{aligned} & \\ & E2 := (-1/2)E2 \\ & E3 := E3 + \tfrac{1}{2}E2 \end{aligned}$$

$$\longrightarrow \left.\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}\right\} \begin{aligned} & E1 := E1 - \tfrac{1}{2}E3 \\ & \\ & E3 := \tfrac{1}{2}E3. \end{aligned}$$

2(b) $\therefore\ c_1 = c_2 = c_3 = 0$. Hence the 3 vectors are linearly independent.

3. $$\begin{bmatrix} 1 & -1 & 0 & 3 \\ -1 & 1 & -1 & -1 \\ 2 & -2 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} R2 := R2 + R1 \\ R3 := R3 - 2R1 \end{matrix}$$

$$\begin{matrix} \uparrow & & \uparrow & \end{matrix}$$

$$\longrightarrow \text{Reduced row echelon form} = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R2 := -R2 \\ R3 := R3 + R2 \end{matrix}$$

(a) The reduced row echelon form has leading 1's only in the 1st & 3rd columns. So the 1st & 3rd columns of the original matrix will be a basis of the column space. So $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis of the column space of the matrix.

(b) To find the nullspace we solve the system $\left.\begin{matrix} x_1 - x_2 + 3x_4 = 0 \\ x_3 - 2x_4 = 0 \end{matrix}\right\}$

So $x_2 = s$, $x_4 = t$, $x_1 = s - 3t$, $x_3 = 2t$.

$\therefore$ Nullspace $= \left\{ \begin{pmatrix} s-3t \\ s \\ 2t \\ t \end{pmatrix} ; s, t \in \mathbb{R} \right\} = \left\{ s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix} ; t \in \mathbb{R} \right\}$

$\therefore \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis of the nullspace.

(c) Another way to find a basis of the nullspace is to insert a new row i with only a "-1" in the (i,i) position (among the non-zero rows of the reduced row echelon form) for each free variable x_i. Then the columns (of the new matrix) without leading 1's will be a basis of the nullspace. Since x_2 & x_4 are free we get

$$\begin{array}{l} \\ \text{new row 2} \\ \\ \text{new row 4} \end{array} \begin{bmatrix} \underline{\underline{1}} & -1 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \underline{\underline{1}} & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{matrix} & \uparrow & & \uparrow \end{matrix}$$

. So $\left\{ \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ -1 \end{pmatrix} \right\}$ is another basis of the null space.

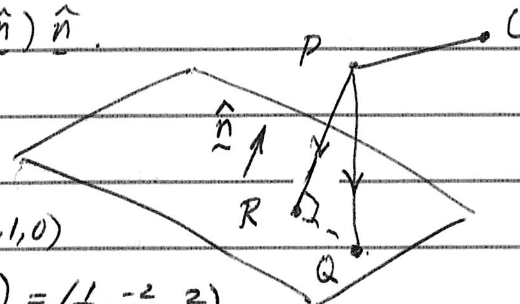
[Leading 1's indicated by double underlines]

"$\uparrow$" – indicates columns without leading 1's.

Note that the new rows we added will be all 0's except for the "-1" in the diagonal positions & the leading 1's will be in the diagonal also.

4(a) Let O be the origin, $P = (2,2,3)$, Q be a point in the plane, and $\underset{\sim}{\hat{n}}$ = the unit normal to the plane and R be the point in the plane nearest to P. Then $PR = (PQ.\underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}}$

So $R = OR = OP + PR = P + (PQ.\underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}}$.



To find Q, just find a solution of $x - 2y + 2z = -2$.

Take $x = 0, y = 1, z = 0$. Then $Q = (0,1,0)$

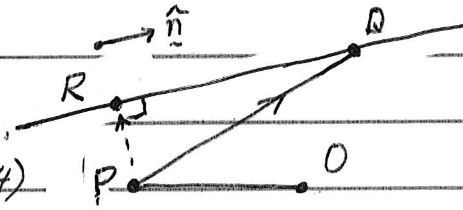
Also $\hat{n} = (1,-2,2)/\sqrt{(1^2+(-2)^2+2^2)} = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

$\therefore\ PQ = PO + OQ = -OP + OQ = -P + Q = -(2,2,3) + (0,1,0) = (-2,-1,-3)$

$$\begin{aligned}\therefore\ R &= (2,2,3) + \left[(-2,-1,-3)\cdot\left(\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3}\right)\right]\left(\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3}\right)\\ &= (2,2,3) + \left(\tfrac{-2}{3} + \tfrac{2}{3} - \tfrac{6}{3}\right)\left(\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3}\right)\\ &= (2,2,3) - 2\left(\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3}\right) = \left(\tfrac{6}{3}, \tfrac{6}{3}, \tfrac{9}{3}\right) - \left(\tfrac{2}{3}, \tfrac{-4}{3}, \tfrac{-4}{3}\right) = \left(\tfrac{4}{3}, \tfrac{10}{3}, \tfrac{5}{3}\right)\end{aligned}$$

Check: $\frac{4}{3} - 2\left(\frac{10}{3}\right) + 2\left(\frac{5}{3}\right) = \frac{4-20+10}{3} = \frac{-6}{3} = -2$, so R is indeed in the plane $x - 2y + 2z = -2$.

(b) Let O be the origin, $P = (3,2,0)$, $Q = (1,0,-4)$, $\hat{n}$ = the unit vector in the direction of the line, and R be the point on the line that is nearest P. Then Q is on the line, $\hat{n} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, and $\|PR\| = \|PQ \times \underset{\sim}{\hat{n}}\|$



Now $PQ = PO + OQ = -OP + OQ$

$= (-3,-2,0) + (1,0,-4) = (-2,-2,-4)$

$$PQ \times \underset{\sim}{\hat{n}} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & -2 & -4 \\ 2/3 & 1/3 & 2/3 \end{vmatrix} = \left\langle -\tfrac{4}{3} + \tfrac{4}{3}, -\tfrac{8}{3} + \tfrac{4}{3}, -\tfrac{2}{3} + \tfrac{4}{3} \right\rangle = \left\langle 0, \tfrac{-4}{3}, \tfrac{2}{3} \right\rangle$$

$\therefore\ \|PR\| = \|PQ \times \hat{n}\| = \sqrt{\left\{0^2 + \frac{16}{9} + \frac{4}{9}\right\}} = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3}$.

So shortest distance between $(3,2,0)$ and the line is $\frac{2\sqrt{5}}{3}$.

5(a) $L(\underset{\sim}{0}_V) = L(0 . \underset{\sim}{0}_V)$ because $0 . \underset{\sim}{x} = \underset{\sim}{0}_V$ for any $\underset{\sim}{x} \in V$

$= 0 . L(\underset{\sim}{0}_V)$ because $L(\alpha . \underset{\sim}{x}) = \alpha . L(\underset{\sim}{x})$ for any $\underset{\sim}{x} \in V$

$= \underset{\sim}{0}_W$ because $0 . \underset{\sim}{u} = \underset{\sim}{0}_W$ for any $\underset{\sim}{u} \in W$.

(b) We know that $\ker(L) = \{\underset{\sim}{x} \in V : L(\underset{\sim}{x}) = \underset{\sim}{0}_W\}$. Since $L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$, it follows that $\underset{\sim}{0}_V \in \ker(L)$. So $\ker(L) \neq \emptyset$.
Now suppose $\alpha \in \mathbb{R}$ and $\underset{\sim}{x} \in \ker(L)$. Then $L(\underset{\sim}{x}) = \underset{\sim}{0}_W$.
So $L(\alpha \underset{\sim}{x}) = \alpha L(\underset{\sim}{x}) = \alpha . \underset{\sim}{0}_W = \underset{\sim}{0}_W$. $\therefore$ $\alpha \underset{\sim}{x} \in \ker(L)$
Also if $\underset{\sim}{x}, \underset{\sim}{y} \in \ker(L)$, then $L(\underset{\sim}{x}) = \underset{\sim}{0}_W$ and $L(\underset{\sim}{y}) = \underset{\sim}{0}_W$.
So $L(\underset{\sim}{x} + \underset{\sim}{y}) = L(\underset{\sim}{x}) + L(\underset{\sim}{y}) = \underset{\sim}{0}_W + \underset{\sim}{0}_W = \underset{\sim}{0}_W$. $\therefore$ $\underset{\sim}{x} + \underset{\sim}{y} \in \ker(L)$
Hence $\ker(L)$ is a subspace of V.

6(a) $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_k\}$ is an orthogonal set of vectors in $\mathbb{R}^n$ if each $\underset{\sim}{v}_i$ is non-zero and $\underset{\sim}{v}_i . \underset{\sim}{v}_j = 0$ whenever $i \neq j$.

(b) Let $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_k\}$ be an orthogonal set of unit vectors in $\mathbb{R}^n$. Now suppose $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}$.
Then by taking the dot product on both sides with $\underset{\sim}{v}_i$ we get $\underset{\sim}{v}_i . (c_1 \underset{\sim}{v}_1 + \cdots + c_k \underset{\sim}{v}_k) = \underset{\sim}{v}_i . \underset{\sim}{0}$. So

$c_i(\underset{\sim}{v}_i . \underset{\sim}{v}_i) = 0$ because $\underset{\sim}{v}_i . \underset{\sim}{v}_j = 0$ whenever $i \neq j$

$\therefore$ $c_i \|\underset{\sim}{v}_i\|^2 = 0 \Rightarrow c_i . 1 = 0$ because $\underset{\sim}{v}_i$ is a unit vector.

$\therefore$ $c_i = 0$ for each i.

Hence $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_k\}$ is a linearly independent set of vectors.