MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #1 - SPRING '01 TIME: 75 min.

Answer all 6 questions. NO CALCULATORS ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(15) 1. Find the solution set of the following system of equations by using Gaussian elimination and back substitution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + 5x_2 - 5x_3 &= 3 \\ -x_1 - x_2 - x_3 &= -4 \end{aligned}$$

(20) 2. Use row operations to transform the augmented matrix of the following system into reduced row echelon form. Then find the solution set of the system.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -1 \\ -x_1 - 4x_2 + 7x_3 &= 3 \\ x_1 + 3x_2 - 5x_3 &= -2 \end{aligned}$$

(15) 3. Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$. Find A^{-1} and check that $AA^{-1}=I$.

(15) 4. (a) Use the cofactor expansion to find $\begin{vmatrix} -2 & 4 & -1 \\ 1 & -2 & 3 \\ 0 & -2 & 3 \end{vmatrix}$

(You may expand along any row or column).

(b) Check your answer by using row operations.

(15) 5. (a) Define what it means for a matrix A to be invertible.

(b) Suppose C and D are invertible nxn matrices. Using only your definition and the properties of matrix algebra, prove that $C^{-1}D$ is also invertible.

(20) 6. (a) Define carefully what is a subspace of a vector space V.

(b) Let A be an nxp matrix and B & C be mxn matrices. Prove that

$$(C+B)A = CA + BA. \quad \text{(Show each step of the proof.)}$$

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SOLUTIONS TO TEST # 1 — SPRING 2001

1.
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + 5x_2 - 5x_3 &= 3 \\ -x_1 - x_2 - x_3 &= -4 \end{aligned} \longrightarrow \begin{aligned} x_1 + 2x_2 - x_3 &= 3 & & \\ x_2 - 3x_3 &= -3 & & E2 := E2 - 2E1 \\ x_2 - 2x_3 &= -1 & & E3 := E3 + E1 \end{aligned}$$

$$\longrightarrow \begin{aligned} x_1 + 2x_2 - x_3 &= 3 & & \\ x_2 - 3x_3 &= -3 & & \\ x_3 &= 2 & & E3 := E3 - E2 \end{aligned}$$

$$\therefore \quad \begin{aligned} x_3 &= 2 \\ x_2 &= 3x_3 - 3 = 3 \\ x_1 &= -2x_2 + x_3 + 3 = -1 \end{aligned}$$

The solution set is: $\left\{ \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right\}$

2.
$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ -1 & -4 & 7 & 3 \\ 1 & 3 & -5 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & -2 & -1 \end{array}\right] \quad \begin{aligned} &R2 := R2 + R1 \\ &R3 := R3 - R1 \end{aligned}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} &R2 := (-1/2) R2 \\ &R3 := R3 + \tfrac{1}{2} R2 \end{aligned}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R1 := R1 - 2R2$$

$$\therefore \quad \begin{aligned} x_3 &= \alpha \\ x_2 &= 2x_3 - 1 = 2\alpha - 1 \\ x_1 &= -x_3 + 1 = 1 - \alpha \end{aligned}$$

Solution set $= \left\{ \begin{pmatrix} 1-\alpha \\ 2\alpha - 1 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$

3. $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$ $R3 := R3 + R1$

$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$ $R1 := R1 - 2R3$, $R2 := R2 - R3$

$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -4 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$ $R1 := R1 + 2R2$

$\therefore A^{-1} = \begin{bmatrix} -3 & 2 & -4 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$. <u>Check</u>: $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 2 & -4 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4. (a) $\begin{vmatrix} -2 & 4 & -1 \\ 1 & -2 & 3 \\ 0 & -2 & 3 \end{vmatrix} = (-2)\cdot(-1)^{1+1}\cdot\begin{vmatrix} -2 & 3 \\ -2 & 3 \end{vmatrix} + 1\cdot(-1)^{2+1}\begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix} + 0$

expanding along column 1

$= (-2).1.0 + 1.(-1).(10) + 0 = -10$.

(b) $\begin{vmatrix} -2 & 4 & -1 \\ 1 & -2 & 3 \\ 0 & -2 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 4 & -1 \\ 0 & 0 & 5/2 \\ 0 & -2 & 3 \end{vmatrix}$ $(R2 := R2 + (1/2)R1)$

$= -\begin{vmatrix} -2 & 4 & -1 \\ 0 & -2 & 3 \\ 0 & 0 & 5/2 \end{vmatrix} = -(-2)(-2)(5/2) = -10.$

$\begin{pmatrix} R2 := R3 \\ R3 := R2 \end{pmatrix}$

5. (a) The matrix A is <u>invertible</u> if we can find a matrix B such that $AB = I$ and $BA = I$.

(b) Consider the matrix $D^{-1}C$. We have

$$(D^{-1}C)(C^{-1}D) = D^{-1}(CC^{-1})D$$
$$= D^{-1}ID = D^{-1}D = I$$

and

$$(C^{-1}D)(D^{-1}C) = C^{-1}(DD^{-1})C$$
$$= C^{-1}IC = C^{-1}C = I$$

So from the definition of invertible, $C^{-1}D$ is invertible.

6. (a) A <u>subspace</u> of the vector space V is any subset S of V such that

1. $S \neq \emptyset$,
2. If $\underline{x} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\underline{x} \in S$
3. If $\underline{x}, \underline{y} \in S$ then $\underline{x}+\underline{y} \in S$.

(b)
$$((C+B)A)[i,j] = \sum_{k=1}^{n} (C+B)[i,k] \cdot A[k,j]$$
$$= \sum_{k=1}^{n} \left(C[i,k] + B[i,k]\right) \cdot A[k,j]$$
$$= \sum_{k=1}^{n} \left(C[i,k] \cdot A[k,j]\right) + \left(B[i,k] \cdot A[k,j]\right)$$
$$= \sum_{k=1}^{n} C[i,k] \cdot A[k,j] + \sum_{k=1}^{n} B[i,k] \cdot A[k,j]$$
$$= (CA)[i,j] + (BA)[i,j]$$
$$= (CA + BA)[i,j]$$

$$\therefore \quad (C+B)A = CA + BA.$$