MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #2 - SPRING '01 TIME: 75 min.

Answer all 6 questions. NO CALCULATORS ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(20) 1. (a) Define what it means for the vectors $\underline{v}_1, \ldots, \underline{v}_n$ to be linearly independent.

(b) Use your definition to determine whether or not the three vectors $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$ are linearly independent.

(20) 2. (a) Define what it means for the vectors $\underline{v}_1, \ldots, \underline{v}_n$ to span R^k.

(b) Use your definition to determine whether or not the following three vectors span R^3: $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}0\\2\\3\end{bmatrix}$.

(15) 3. Find a basis for the row space and a basis for the null space of the matrix

$$\begin{bmatrix}1 & -2 & 1\\-2 & 4 & -3\\1 & -2 & 2\end{bmatrix}.$$

(15) 4. Let $\underline{u}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\underline{v}_1 = \begin{bmatrix}2\\3\end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix}3\\4\end{bmatrix}$.

(a) Find the transition matrix from the basis $[\underline{u}_1, \underline{u}_2]$ to the standard basis $[\underline{e}_1, \underline{e}_2]$ and the transition matrix from $[\underline{e}_1, \underline{e}_2]$ to the basis $[\underline{v}_1, \underline{v}_2]$.

(b) Find the transition matrix from the basis $[\underline{v}_1, \underline{v}_2]$ to the basis $[\underline{u}_1, \underline{u}_2]$.

(15) 5. (a) Define what is a linear transformation from the vector space V to the vector space W.

(b) If L is a linear transformation from V to W and $\underline{x}, \underline{z} \in V$, prove that $L(\underline{0}_V) = \underline{0}_W$ and $L(3\underline{x}-2\underline{z}) = 3L(\underline{x})-2L(\underline{z})$.

(15) 6. Let A be any mxn matrix. Explain why we always have

$$\text{rank}(A) + \text{nullity}(A) = n.$$

(You may use any theorems given in class, except the rank & nullity theorem, to support your argument.)

MAS 3105 - Linear Algebra — Florida Int'l Univ.

Solutions to Test # 2 — Spring 2001

1(a) The vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n$ are linearly independent if the equation $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_n \underset{\sim}{v}_n = \underset{\sim}{0}$ implies that $c_1 = c_2 = \cdots = c_n = 0$.

(b) Suppose $c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then

$$\left.\begin{array}{rcl} c_1 \qquad + 2c_3 &=& 0 \\ c_2 + c_3 &=& 0 \\ -c_1 + c_2 + c_3 &=& 0 \end{array}\right\} \longrightarrow \left.\begin{array}{rcl} c_1 \qquad + 2c_3 &=& 0 \\ c_2 + c_3 &=& 0 \\ 2c_2 + 3c_3 &=& 0 \end{array}\right\} \quad E3 := E3 + E1$$

$$\longrightarrow \left.\begin{array}{rcl} c_1 \qquad + 2c_3 &=& 0 \\ c_2 + c_3 &=& 0 \\ c_3 &=& 0 \end{array}\right\} \quad E3 := E3 - 2E2$$

$$\longrightarrow \left.\begin{array}{rcl} c_1 &=& 0 \\ c_2 &=& 0 \\ c_3 &=& 0 \end{array}\right\} \quad \begin{array}{l} E1 := E1 - 2E3 \\ E2 := E2 - E3 \end{array}$$

$\therefore$ $c_1 = c_2 = c_3 = 0$. Hence the 3 vectors are linearly independent.

2(a) The vectors $\underset{\sim}{v}_1, \cdots, \underset{\sim}{v}_n$ span $\mathbb{R}^k$ if every vector $\underset{\sim}{v} \in \mathbb{R}^k$ can be expressed as $\underset{\sim}{v} = \alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \cdots + \alpha_n \underset{\sim}{v}_n$ with $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$.

(b) We will try to see if every vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ can be expressed as a linear combination of the three given vectors.

Suppose $\alpha_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then

$$\left.\begin{array}{rcl} \alpha_1 + 2\alpha_2 \qquad &=& a \\ 2\alpha_2 + 2\alpha_3 &=& b \\ -\alpha_1 + \alpha_2 + 3\alpha_3 &=& c \end{array}\right\} \longrightarrow \left.\begin{array}{rcl} \alpha_1 + 2\alpha_2 \qquad &=& a \\ 2\alpha_2 + 2\alpha_3 &=& b \\ 3\alpha_2 + 3\alpha_3 &=& c + a \end{array}\right\} \quad E3 := E3 + E1$$

$$\longrightarrow \left.\begin{array}{rcl} \alpha_1 + 2\alpha_2 \qquad &=& a \\ 2\alpha_2 + 2\alpha_3 &=& b \\ 0 &=& c + a - \frac{3b}{2} \end{array}\right\} \quad E3 := E3 - \tfrac{3}{2}E2$$

2(b) So if we take $a=1$, $b=0$, & $c=0$ the system will be inconsistent. Hence $\begin{pmatrix}1\\0\\0\end{pmatrix}$ cannot be expressed as a lin. comb. of the 3 vectors. So the 3 vectors do not span $\mathbb{R}^3$.

3. $$\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -3\\ 1 & -2 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 1\\ 0 & 0 & -1\\ 0 & 0 & 1\end{bmatrix} \quad \begin{matrix}R2 := R2+2R1\\ R3 := R3-R1\end{matrix}$$

$$\sim (\text{Row echelon form}) = \begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \quad \begin{matrix}R2 := (-1)R2\\ R3 := R3+R2\end{matrix}$$

$$\sim (\text{Reduced row echelon form}) = \begin{bmatrix}1 & -2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \quad R1 := R1-R2$$

(a) The non-zero rows of the row echelon form will give us a basis for the row space of the matrix. So $\{(1,-2,1), (0,0,1)\}$ is a basis of the row space. [We can also use the non-zero rows of the Reduced row echelon form to get another basis $\{(1,-2,0),(0,0,1)\}$ of the row space.]

(b) We found the reduced row echelon form because we need it to get the null space. We have $\left.\begin{matrix}x_1 - 2x_2 = 0\\ x_3 = 0\end{matrix}\right\}$

So $x_2 = t$, $x_1 = 2x_2 = 2t$, and $x_3 = 0$

$\therefore$ Nullspace $= \left\{\begin{pmatrix}2t\\t\\0\end{pmatrix} : t \in \mathbb{R}\right\} = \left\{t\begin{pmatrix}2\\1\\0\end{pmatrix} : t \in \mathbb{R}\right\}$. So $\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}\right\}$ is a basis of the nullspace.

4 (a) Transition matrix from $[\underset{\sim}{u}_1, \underset{\sim}{u}_2]$ to $[\underset{\sim}{e}_1, \underset{\sim}{e}_2]$ $= [\underset{\sim}{u}_1 \ \underset{\sim}{u}_2] = \begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$.

Transition matrix from $[\underset{\sim}{e}_1, \underset{\sim}{e}_2]$ to $[\underset{\sim}{v}_1, \underset{\sim}{v}_2]$ $= [\underset{\sim}{v}_1 \ \underset{\sim}{v}_2]^{-1}$

$$= \begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}^{-1} = \frac{1}{2(4)-3(3)}\begin{bmatrix}4 & -3\\ -3 & 2\end{bmatrix} = \begin{bmatrix}-4 & 3\\ 3 & -2\end{bmatrix}$$

(b) Transition matrix from $[\underset{\sim}{u}_1, \underset{\sim}{u}_2]$ to $[\underset{\sim}{v}_1, \underset{\sim}{v}_2]$ $= \begin{bmatrix}-4 & 3\\ 3 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$.

4(b) Transition matrix from $[\underset{\sim}{v}_1, \underset{\sim}{v}_2]$ to $[\underset{\sim}{u}_1, \underset{\sim}{u}_2] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2-1}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

5(a) A linear transformation from V to W is a function $L: V \to W$ such that $L(\alpha \underset{\sim}{x}) = \alpha . L(\underset{\sim}{x})$ & $L(\underset{\sim}{x}+\underset{\sim}{y}) = L(\underset{\sim}{x}) + L(\underset{\sim}{y})$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in V$ & $\alpha \in \mathbb{R}$.

(b) $L(\underset{\sim}{0}_V) = L(0.\underset{\sim}{0}_V)$ because $0.\underset{\sim}{x} = \underset{\sim}{0}_V$ for any vector $\underset{\sim}{x}$ in V

$= 0.\ L(0_V)$ because $L(\alpha \underset{\sim}{x}) = \alpha L(\underset{\sim}{x})$

$= \underset{\sim}{0}_W$ because $0.\underset{\sim}{u} = \underset{\sim}{0}_W$ for any vector $\underset{\sim}{u}$ in W

$L(3\underset{\sim}{x} - 2\underset{\sim}{z}) = L[3\underset{\sim}{x} + (-2)\underset{\sim}{z}] = L(3\underset{\sim}{x}) + L[(-2)\underset{\sim}{z}]$

$= 3L(\underset{\sim}{x}) + (-2)L(\underset{\sim}{z}) = 3L(\underset{\sim}{x}) - [2L(\underset{\sim}{z})]$.

6. If we transform A into its reduced row echelon form U_R, then the non-zero rows of U_R will span the same subspace of $\mathbb{R}^n$ that the rows of A spanned - because they were obtained from the latter by row operations. Also the non-zero rows of U_R are linearly independent because each of these non-zero rows had a different number of leading zeros. So the non-zero rows of U_R will form a basis of the row space $R(A)$ of A. Let $r = \text{rank}(A)$. Then by definition $r = \dim[R(A)]$. So $r =$ no. of non-zero rows of U_R = no. of leading 1's in U_R.

Now in U_R, the leading 1's will be in the columns which correspond to the r bound variables. The other $n-r$ variables will be free. So when we solve the system $U_R\underset{\sim}{x} = \underset{\sim}{0}$ to get the nullspace $N(A)$ of A, we will have $n-r$ free variables and this will show us that the nullspace has dimension $n-r$. To get a basis of $N(A)$ set one of the free variable equal to -1 and the rest of the other free variables equal to 0. (The bound variables will get their values from the one we set equal to -1.) If we do this for each free variable we will get $n-r$ vectors which form a basis of $N(A)$. So

$$\text{rank}(A) + \text{nullity}(A) = \dim R(A) + \dim N(A) = r + (n-r) = n.$$