MAS 3105 - LINEAR ALGEBRA | FLORIDA INT'L UNIV.

TEST #1 - SPRING 2014 | TIME: 75 min.

Answer all 6 questions. NO CALCULATORS or CELL PHONES ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(20) 1(a) Use row operations to transform the ***augmented matrix*** of the following system into ***reduced row echelon form***.

(b) Then find the ***solution set*** of the system in standard form.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 &= 1 \\ x_1 + 2x_2 - x_3 - 2x_4 &= -3 \\ -x_1 - 2x_2 + 2x_3 + 0x_4 &= 1 \end{aligned}$$

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(20) 2(a) Define what is an ***invertible matrix***. Let $A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & -2 & -5 \\ -1 & 1 & 4 \end{bmatrix}$.

(b) Find A^{-1} by using ***row operations*** and verify that $AA^{-1} = I_3$.

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(15) 3(a) Let $B = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 4 & 1 \\ -1 & 2 & -3 \end{bmatrix}$. Find det(B) by using ***row operations***.

(b) Check your answer by using the ***Laplace's cofactor expansion***.

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(15) 4(a) Define what is the ***span*** of the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$.

(b) Let $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, & $\underline{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Is $\underline{u} \in \mathit{span}\{\underline{v}_1, \underline{v}_2\}$?

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(15) 5(a) Define what is a ***subspace*** of the vector space $\langle V, +, \cdot, -, \underline{0} \rangle$.

(b) If B and C are mxn matrices and A is an nxp matrix, prove that $(B+C)A = BA + CA$.

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(15) 6(a) Let A be an nxn matrix. Define what are the ***(i,j) minor matrix***, M_{ij}, and the ***(i,j) cofactor***, A_{ij}, of A.

(b) Let A' be the matrix obtained by interchanging row 1 & row 2 of A. Prove by induction that $\det(A') = -\det(A)$, for each $n \geq 2$. *[You may use the Laplace cofactor expansion of det(A) without proof, if needed.]*

MAS 3105 - Linear Algebra   Florida International Univ.

Solutions to Test #1   Spring 2014

1(a)

$$\underbrace{\begin{bmatrix} 1 & 2 & -3 & 2 \\ 1 & 2 & -1 & -2 \\ -1 & -2 & 2 & 0 \end{bmatrix}}_{A} \left|\, \underbrace{\begin{matrix} 1 \\ -3 \\ 1 \end{matrix}}_{\underset{\sim}{b}} \right. \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 1 \\ 0 & 0 & 2 & -4 & -4 \\ 0 & 0 & -1 & 2 & 2 \end{array}\right] \begin{array}{l} \\ R2 := R2 - R1 \\ R3 := R3 + R1 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 2 & 2 \end{array}\right] \begin{array}{l} \\ R2 := \frac{1}{2} R2 \\ \\ \end{array}$$

Reduced row echelon form of Augmented matrix

$$\longrightarrow \left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & -4 & -5 \\ 0 & 0 & \underline{1} & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1 := R1 + 3R2 \\ \\ R2 := R3 + R2 \end{array}$$

$$\underbrace{}_{A_R} \quad \underbrace{}_{\underset{\sim}{b}_R}$$

(b) Insert or delete rows of zeros in A_R to get a square matrix A_s with the leading 1's in the diagonal.

$$\longrightarrow \left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{insert row of 0's} \\ \\ \\ \end{array}$$

$$\underbrace{}_{A_s} \quad \underbrace{}_{\underset{\sim}{b}_s}$$

$$(I - A_s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Solution set, $S = \underset{\sim}{b}_s + \text{span}(\text{non-zero columns of } I - A_s)$

$$= \left\{ \begin{pmatrix} -5 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

Alt. (b) solution

$$\left.\begin{array}{r} x_1 + 2x_2 - 4x_4 = -5 \\ x_3 - 2x_4 = -2 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -5 - 2\alpha + 4\beta, \\ x_3 = -2 + 2\beta \end{array} \quad S = \left\{ \begin{pmatrix} -5 - 2\alpha + 4\beta \\ \alpha \\ -2 + 2\beta \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} =$$

$x_2 = \alpha, \ x_4 = \beta$

$$= \left\{ \begin{pmatrix} -5 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

2 (a) A is invertible if we can find a matrix B such that $AB = I_n = BA$.

(b)
$$\underbrace{\left[\begin{array}{ccc} 1 & -1 & -3 \\ 1 & -2 & -5 \\ -1 & 1 & 4 \end{array}\right.}_{A} \underbrace{\left|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{I_3} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R2 := R2 - R1 \\ R3 := R3 + R1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \quad R2 := -R2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \quad R1 := R1 + R2$$

$$\rightarrow \underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I_3} \underbrace{\left|\begin{array}{ccc} 3 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{array}\right]}_{A^{-1}} \begin{array}{l} R1 := R1 + R3 \\ R2 := R2 - 2R3 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}. \quad AA^{-1} = \begin{bmatrix} 1 & -1 & -3 \\ 1 & -2 & -5 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

3 (a)
$$\begin{vmatrix} -1 & 0 & -2 \\ 2 & 4 & 1 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & -2 \\ 0 & 4 & -3 \\ 0 & 2 & -1 \end{vmatrix} \begin{pmatrix} R2 := R2 - 2R1 \\ R3 := R3 - R1 \end{pmatrix} = \begin{vmatrix} -1 & 0 & -2 \\ 0 & 4 & -3 \\ 0 & 0 & 1/2 \end{vmatrix} \left(R3 := R3 - \tfrac{1}{2} R2 \right)$$

$$= (-1)(4)(1/2) = -2.$$

(b)
$$\begin{vmatrix} -1 & 0 & -2 \\ 2 & 4 & 1 \\ -1 & 2 & -3 \end{vmatrix} = (-1) \cdot (-1)^{1+1} \cdot \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} + 0 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} + (-2) \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= (-1)(-12-2) + 0 + (-2)(4-(-4))$$

$$= (-1)(-14) + (-2)(8) = 14 - 16 = -2.$$

4(a) $\text{Span}(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k) = \{\alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \cdots + \alpha_k \underset{\sim}{v}_k : \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}$.

(b) Suppose $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 = \underset{\sim}{u}$. Then $c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$$\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 + 2c_2 = -2 \\ -2c_1 - 3c_2 = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} c_1 + c_2 = 1 \\ c_2 = -3 \\ -c_2 = 3 \end{array}\right\} \begin{array}{l} E2 := E2 - E1 \\ E3 := E3 + 2E \end{array} \Rightarrow \left.\begin{array}{l} c_1 = 4 \\ c_2 = -3 \\ 0 = 0 \end{array}\right\} \begin{array}{l} E1 := E1 - E2 \\ E3 := E3 + E2 \end{array}$$

$\therefore \underset{\sim}{u} = 4\underset{\sim}{v}_1 + (-3)\underset{\sim}{v}_2$. So $\underset{\sim}{u} \in \text{span}(\underset{\sim}{v}_1, \underset{\sim}{v}_2)$.

5(a) A subspace of V is any subset S of V which satisfies (i) $S \neq \emptyset$, (ii) $\underset{\sim}{x}, \underset{\sim}{y} \in S \Rightarrow \underset{\sim}{x} + \underset{\sim}{y} \in S$, and (iii) $\alpha \in \mathbb{R}$ & $\underset{\sim}{x} \in S \Rightarrow \alpha \underset{\sim}{x} \in S$.

(b) $\{(B+C)A\}[i,j] = \sum_{k=1}^{n} (B+C)[i,k] \cdot A[k,j] = \sum_{k=1}^{n} \{B[i,k] + C[i,k]\} \cdot A[k,j]$

$= \sum_{k=1}^{n} \{B[i,k] \cdot A[k,j] + C[i,k] \cdot A[k,j]\} = \sum_{k=1}^{n} B[i,k] \cdot A[k,j] + \sum_{k=1}^{n} C[i,k] \cdot A[k,j]$

$= (BA)[i,j] + (CA)[i,j] = (BA + CA)[i,j]$. $\therefore (B+C)A = BA + CA$.

6(a) M_{ij} = the $(n-1)\times(n-1)$ matrix obtained by deleting row i & column j from A. $A_{ij} = (-1)^{i+j} \det(M_{ij})$

(b) <u>Basis</u>: If $n = 2$, then $\det(A') = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$

$= -(a_{11}a_{22} - a_{12}a_{21}) = -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -\det(A)$.

where $n \geq 3$

<u>Ind. Step</u>: Suppose the result is true for all $(n-1)\times(n-1)$-matrices. Then by expanding $\det(A')$ along row 3, we get

$\det(A') = a_{31} \cdot (-1)^{3+1} \cdot \det[M_{31}(A')] + a_{32} \cdot (-1)^{3+2} \cdot \det[M_{32}(A')] + \cdots + a_{3n} \cdot (-1)^{3+n} \cdot \det[M_{3n}(A')]$

$= -a_{31} \cdot (-1)^{3+1} \cdot \det[M_{31}(A)] - a_{32} \cdot (-1)^{3+2} \cdot \det[M_{32}(A)] - \cdots - a_{3n}(-1)^{3+n} \cdot \det[M_{3n}(A)]$

$= -(a_{31}A_{31} + a_{32}A_{32} + \cdots + a_{3n}A_{3n}) = -\det(A)$. So if the result is true for $n-1$, it will be true for n.

<u>Conclusion</u> Hence by the Principle of Mathematical Induction, $\det(A') = -\det(A)$ for each $n \geq 2$.