MAS 3305 - LINEAR ALGEBRA FLORIDA INT'L UNIV.

TEST #2 - SPRING 2014 TIME: 75 min.

Answer all 6 questions. NO CALCULATORS or CELL PHONES ARE ALLOWED. Show all working and provide all reasoning where required. An unjustified answer will receive little or no credit. ***Begin each of the 6 questions on 6 separate pages.***

(15) 1. Let $E = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle$, $G = \langle \begin{pmatrix}-3\\5\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} \rangle$, and $H = \langle \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix} \rangle$.

(a) Find the *transition matrix*, ${}_E(I)_G$, from the basis G to the basis E; and the *transition matrix*, ${}_H(I)_E$, from E to H.

(b) Find the *transition matrix*, ${}_H(I)_G$, from the basis G to the basis H; and the *transition matrix*, ${}_G(I)_H$, from H to G.

(25) 2. Let $A = \begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & -4 & 5 & 1 \\ -1 & 2 & -4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \\ 1 & -1 & -2 \end{pmatrix}$.

(a) Find bases for the *row space* and the *null space* of the matrix A.

(b) Find bases for the *co-null space* and the *column space* of the matrix B.

(15) 3. Find the *best least squares fit* by a *linear function* $c_0 + c_1.x$ to the data below and then *check how close* your answers are to $y(x)$.

x	0	1	3
$y(x)$	2	1	0

x	0	1	3
$c_0 + c_1.x$	?	?	?

(15) 4. Using the *Gram-Schmidt orthogonalization process*, find an *orthonormal basis* for the subspace $S = \text{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}6\\6\\0\end{pmatrix} \right\}$ and check it is orthonormal.

(15) 5(a) Define what is a *linear transformation* from the vector space V to the vector space W.

(b) If L is a *linear transformation* from V to W and R is a *subspace* of W, prove that $L^{-1}[R]$ is a *subspace* of V.

(15) 6(a) Define what is an *orthogonal set* of vectors in $\mathbb{R}^n$ and what is the *orthogonal complement* $U^\perp$ of the subspace U of $\mathbb{R}^n$.

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an *orthogonal set* of *non-zero* vectors in $\mathbb{R}^n$, prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is *linearly independent.*

MAS 3105 – Linear Algebra   Florida International Univ.

Solutions to Test #2   Spring 2014

1(a) ${}_E(I)_G = \begin{bmatrix}-3&-1\\5&2\end{bmatrix}$ ✓ ${}_G(I)_E = \begin{bmatrix}-3&-1\\5&2\end{bmatrix}^{-1} = \frac{1}{-6+5}\begin{bmatrix}2&1\\-5&-3\end{bmatrix} = \begin{bmatrix}-2&-1\\5&3\end{bmatrix}$

${}_H(I)_E = \left[{}_E(I)_H\right]^{-1} = \begin{bmatrix}1&1\\3&4\end{bmatrix}^{-1} = \frac{1}{4-3}\begin{bmatrix}4&-1\\-3&1\end{bmatrix} = \begin{bmatrix}4&-1\\-3&1\end{bmatrix}$ ✓

(b) ${}_H(I)_G = {}_H(I)_E \cdot {}_G(I)_G = \begin{bmatrix}4&-1\\-3&1\end{bmatrix}\begin{bmatrix}-3&-1\\5&2\end{bmatrix} = \begin{bmatrix}-17&-6\\14&5\end{bmatrix}$ ✓

${}_G(I)_H = {}_G(I)_E\,{}_E(I)_H = \begin{bmatrix}-2&-1\\5&3\end{bmatrix}\begin{bmatrix}1&1\\3&4\end{bmatrix} = \begin{bmatrix}-5&-6\\14&17\end{bmatrix}$ ✓

<u>Check:</u> ${}_G(I)_H = \left[{}_H(I)_G\right]^{-1} = \begin{bmatrix}-17&-6\\14&5\end{bmatrix}^{-1} = \frac{1}{-85+84}\begin{bmatrix}5&6\\-14&-17\end{bmatrix} = \begin{bmatrix}-5&-6\\14&17\end{bmatrix}$

2(a) $\begin{bmatrix}1&-2&2&1\\2&-4&5&1\\-1&2&-4&1\end{bmatrix} \to \begin{bmatrix}1&-2&2&1\\0&0&1&-1\\0&0&-2&2\end{bmatrix}$ $R2 := R2-2R1$, $R3 := R3+R1$

$\to \begin{bmatrix}1&-2&0&3\\0&0&1&-1\\0&0&0&0\end{bmatrix}$ $R1 := R1-2R2$, $R3 := R3+R2$ $\Rightarrow A_s = \begin{bmatrix}1&-2&0&3\\0&0&0&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}$

$I_4 - A_s = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \begin{bmatrix}1&-2&0&3\\0&0&0&0\\0&0&1&-1\\0&0&0&0\end{bmatrix} = \begin{bmatrix}0&2&0&-3\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{bmatrix}$

Basis for Rowspace $= \{(1,-2,0,3),\ (0,0,1,-1)\}$

Basis for nullspace $= \left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\1\end{pmatrix}\right\}$

(b) $\left[\begin{array}{ccc|cccc}1&-1&3&1&0&0&0\\-1&1&-2&0&1&0&0\\2&-2&4&0&0&1&0\\1&-1&-2&0&0&0&1\end{array}\right] \to \left[\begin{array}{ccc|cccc}1&-1&3&1&0&0&0\\0&0&1&1&1&0&0\\0&0&-2&-2&0&1&0\\0&0&-5&-1&0&0&1\end{array}\right]$ $R2 := R2+R1$, $R3 := R3-2R1$, $R4 := R4-R1$

$\to \left[\begin{array}{ccc|cccc}1&-1&3&1&0&0&0\\0&0&1&1&1&0&0\\0&0&0&0&2&1&0\\0&0&0&4&5&0&1\end{array}\right]$ $R3 := R3+2R2$, $R4 := R4+5R2$

Basis for Column Space $= \left\{\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\-2\\4\\-2\end{pmatrix}\right\}$

2(b) Basis for co-null space $= \{(0,2,1,0), (4,5,0,1)\}$

3(a) Suppose $Y(x) = c_0 + c_1 x$. Then $\begin{cases} c_0 + c_1 . 0 = 2 \\ c_0 + c_1 . 1 = 1 \\ c_0 + c_1 . 3 = 0 \end{cases}$. So

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{b}$$

. Normal System is $(A^T A)\hat{\underset{\sim}{c}} = A^T \underset{\sim}{b}$. Therefore we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{30-16} \begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 26 \\ -9 \end{bmatrix} = \begin{bmatrix} 13/7 \\ -9/14 \end{bmatrix}$$

(b) Check:

x	0	1	3
$\hat{c}_0 + \hat{c}_1 x$	$\frac{13}{7}$	$\frac{17}{14}$	$\frac{-1}{14}$

& this is close to

x	0	1	3
$Y(x)$	2	1	0

4(a) $\underset{\sim}{v}_1 = x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underset{\sim}{v}_2 = x_2 - \left(\frac{x_2 \cdot v_1}{\|v_1\|^2} \right) \underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{\|\underset{\sim}{v}_1\|^2} \right) \underset{\sim}{v}_1 - \left(\frac{x_3 \cdot v_2}{\|\underset{\sim}{v}_2\|^2} \right) \underset{\sim}{v}_2 = \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} - \frac{12}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{-6}{2} \right) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Now put $\underset{\sim}{u}_i = \frac{\underset{\sim}{v}_i}{\|\underset{\sim}{v}_i\|}$.

Orthonormal basis is: $\underset{\sim}{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underset{\sim}{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\underset{\sim}{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

$\{v_1, v_2, v_3\}$ is an orthogonal basis.

Check: $\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = -1 + 0 + 1 = 0.$, $\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_3 = (-1)(-1) + 0 + (-1)(1) = 0$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_3 = (-1) + (2) + (-1) = 0$ ✓

5(a) A linear transformation from V to W is any function $L: V \to W$ such that $L(\underset{\sim}{x} + \underset{\sim}{y}) = L(\underset{\sim}{x}) + L(\underset{\sim}{y})$ & $L(\alpha \underset{\sim}{x}) = \alpha L(\underset{\sim}{x})$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in V$ & $\alpha \in \mathbb{R}$.

(b) Let $L: V \to W$ be a linear transformation & R be a subspace of W. Then $L^{-1}[R] = \{\underset{\sim}{x} \in V : L(\underset{\sim}{x}) \in R\}$. Now $L(\underset{\sim}{0}_V) = L(0 . \underset{\sim}{0}_V) = 0 . L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$. So $L(\underset{\sim}{0}_V) = \underset{\sim}{0}_W \in R$ because R is a subspace

$\therefore\ \underset{\sim}{0}_V \in L^{-1}[R]$ because $L(\underset{\sim}{0}_V) \in R$. So (i) $L^{-1}[R] \neq \emptyset$.

Now suppose $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in L^{-1}[R]$ and $\alpha \in \mathbb{R}$. Then $L(\underset{\sim}{x}_1) \in R$ and $L(\underset{\sim}{x}_2) \in R$. So

$$L(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) = L(\underset{\sim}{x}_1) + L(\underset{\sim}{x}_2) \quad \text{bec. } L \text{ is a lin. transf.}$$
$$\in R \quad \text{because } R \text{ is a subspace}$$

$\therefore$ (ii) $\underset{\sim}{x}_1 + \underset{\sim}{x}_2 \in L^{-1}[R]$ because $L(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) \in R$.

Also $L(\alpha \underset{\sim}{x}_1) = \alpha L(\underset{\sim}{x}_1)$ bec. L is a lin. transf.
$$\in R \quad \text{because } R \text{ is a subspace}$$

$\therefore$ (iii) $\alpha \underset{\sim}{x}_1 \in L^{-1}[R]$ because $L(\alpha \underset{\sim}{x}_1) \in R$. Hence $L^{-1}[R]$ is a subspace of V, because it satisfies conditions (i), (ii) & (iii).

6 (a) An <u>orthogonal set</u> of vectors in $\mathbb{R}^n$ is any set S of vectors in $\mathbb{R}^n$ such that $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ for all $\underset{\sim}{u}, \underset{\sim}{v} \in S$ with $\underset{\sim}{u} \neq \underset{\sim}{v}$. The <u>orthogonal complement</u> of $\mathcal{U}$ is defined by $\mathcal{U}^{\perp} = \{ \underset{\sim}{x} \in \mathbb{R}^n : \underset{\sim}{x} \cdot \underset{\sim}{y} = 0 \text{ for each } \underset{\sim}{y} \in \mathcal{U} \}$.

(b) Suppose $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then $\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j = 0$ for all $i \neq j$. Now suppose $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \dots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}$. Then for each i

$$(c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \dots + c_k \underset{\sim}{v}_k) \cdot \underset{\sim}{v}_i = \underset{\sim}{0} \cdot \underset{\sim}{v}_i$$

So $\quad c_1(0) + c_2(0) + \dots + c_i(\underset{\sim}{v}_i \cdot \underset{\sim}{v}_i) + 0 + \dots + c_k(0) = 0$

$\therefore\ c_i \|\underset{\sim}{v}_i\|^2 = 0$. Since each $\underset{\sim}{v}_i$ is non-zero, it follows that $\|\underset{\sim}{v}_i\|^2 \neq 0$. Thus $c_i = 0$ for each $i = 1, \dots, k$.

Hence $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \dots + c_k \underset{\sim}{v}_k = \underset{\sim}{0} \Rightarrow c_1 = c_2 = \dots = c_k = 0$.

So $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k\}$ is linearly independent.