MAS 3105 - LINEAR ALGEBRA | FLORIDA INT'L UNIV.
TEST #1 - SPRING 2016 | TIME: 75 min.

Answer all 6 questions. NO CALCULATORS or CELL PHONES ARE ALLOWED. Show all working in problems 1-4. Provide all reasoning in problems 5-6. An unjustified answer will receive little or no credit.

(20) 1 (a) Use row operations to transform the *augmented matrix* of the following system of linear equations into *reduced row echelon form.*

(b) Then find the *solution set* of the system in standard form.

$$\begin{aligned} -x_1 - 2x_2 + 2x_3 + 0x_4 &= 1 \\ x_1 + 2x_2 - 3x_3 + 2x_4 &= 1 \\ 2x_1 + 4x_2 - 2x_3 - 4x_4 &= -6 \end{aligned}$$

(20) 2 (a) Define what it means for A to be an *invertible matrix.*

(b) Let $A = \begin{bmatrix} 1 & -5 & -2 \\ -1 & 4 & 1 \\ -1 & 3 & 1 \end{bmatrix}$. Find A^{-1} by using *row operations* & verify that $AA^{-1} = I$.

(15) 3 (a) Let $B = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 3 & -2 & 3 \end{bmatrix}$. Find *det(B)* by using *row operations.*

(b) Check your answer by using the *Laplace's cofactor expansion.*

(15) 4 (a) Define what it means for the set, $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$, of vectors to be *linearly independent.*

(b) If $\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, and $\underline{v}_3 = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}$; is $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ *linearly independent ?*

(15) 5 (a) Define what it means for a set, S, of vectors in $\mathbb{R}^n$ to be a *subspace* of $\mathbb{R}^n$.

(b) If A and B are mxn matrices and C is an nxp matrix, prove that $(A+B)C = AC + BC$.

(15) 6 (a) Define what are the *(i,j) minor,* M_{ij}, and the *(i,j) cofactor,* A_{ij}, of an nxn matrix A.

(b) Let A be an nxn matrix and A' be the matrix obtained by interchanging row 1 & row 2 of A. Prove, by induction, that $\det(A') = -\det(A)$, for each $n \geq 2$.

[*You may use the Laplace cofactor expansion of det(A) without proof, if needed.*]

MAS 3105 - Linear Algebra Florida International Univ.
Solutions to Test #1 Spring 2016

1(a)
$$\underbrace{\begin{bmatrix} -1 & -2 & 2 & 0 \\ 1 & 2 & -3 & 2 \\ 2 & 4 & -2 & -4 \end{bmatrix}}_{A} \left| \underbrace{\begin{matrix} 1 \\ 1 \\ -6 \end{matrix}}_{\underset{\sim}{b}} \right. \to \left[\begin{array}{cccc|c} 1 & 2 & -2 & 0 & -1 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 2 & -4 & -4 \end{array}\right] \quad \begin{matrix} R1 := (-1)R1 \\ R2 := R2 + R1 \\ R3 := R3 + 2R1 \end{matrix}$$

$$\to \left[\underbrace{\begin{matrix} \underline{1} & 2 & 0 & -4 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{matrix}}_{A_R} \left| \underbrace{\begin{matrix} -5 \\ -2 \\ 0 \end{matrix}}_{\underset{\sim}{b}_R} \right.\right] \quad \begin{matrix} R1 := R1 - 2R2 \\ R2 := (-1)R2 \\ R3 := R3 + 2R2 \end{matrix}$$

(b) Insert or delete rows of zeros in $[A_R | \underset{\sim}{b}_R]$ to get $[A_s | \underset{\sim}{b}_s]$, where A_s is a square matrix with the leading 1's in the diagonal.

$$\begin{array}{r} \\ \text{row of zeros} \to \\ \\ \\ \end{array} \to \left[\underbrace{\begin{matrix} \underline{1} & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{matrix}}_{A_s} \left| \underbrace{\begin{matrix} -5 \\ 0 \\ -2 \\ 0 \end{matrix}}_{\underset{\sim}{b}_s} \right.\right]. \qquad (I_4 - A_s) = \begin{bmatrix} 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution set $= \underset{\sim}{b}_s$ + span (non-zero columns of $I - A_s$)

$$= \left\{ \begin{pmatrix} -5 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} ; \ \alpha, \beta \in \mathbb{R} \right\}.$$

2(a) A is an invertible matrix if there exists a matrix B such that $AB = I = BA$.

(b)
$$\left[\underbrace{\begin{matrix} 1 & -5 & -2 \\ -1 & 4 & 1 \\ -1 & 3 & 1 \end{matrix}}_{A} \left| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \right.\right] \to \left[\begin{array}{ccc|ccc} 1 & -5 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 & 1 \end{array}\right] \quad \begin{matrix} \\ R2 := R2 + R1 \\ R3 := R3 + R1 \end{matrix}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -4 & -5 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right] \quad \begin{matrix} R1 := R1 - 5R2 \\ R2 := (-1)R2 \\ R3 := R3 - 2R2 \end{matrix}$$

2(b) $\rightarrow \left[\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I_3}\right|\left.\underbrace{\begin{matrix}-1 & 1 & -3\\ 0 & 1 & -1\\ -1 & -2 & 1\end{matrix}}_{A^{-1}}\right]$ $R1:=R1-3R3$, $R2:=R2-R3$ $\therefore A^{-1}=\begin{bmatrix}-1 & 1 & -3\\ 0 & 1 & -1\\ -1 & -2 & 1\end{bmatrix}$.

<u>Check:</u> $AA^{-1}=\begin{bmatrix}1 & -5 & -2\\ -1 & 4 & 1\\ -1 & 3 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & -3\\ 0 & 1 & -1\\ -1 & -2 & 1\end{bmatrix}=\begin{bmatrix}-1+2 & 1-5+4 & -3+5-2\\ 1-1 & -1+4-2 & 3-4+1\\ 1-1 & -1+3-2 & 3-3+1\end{bmatrix}=\overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}^{I_3}$

3(a) $\det(B)=\begin{vmatrix}2 & 1 & 4\\ 1 & 1 & 2\\ 3 & -2 & 3\end{vmatrix}=-\begin{vmatrix}1 & 1 & 2\\ 2 & 1 & 4\\ 3 & -2 & 3\end{vmatrix}$ $\begin{pmatrix}R1:=R2\\ R2:=R1\end{pmatrix}$

$=-\begin{vmatrix}1 & 1 & 2\\ 0 & -1 & 0\\ 0 & -5 & -3\end{vmatrix}$ $\begin{pmatrix}R2:=R2-2R1\\ R3:=R3-3R1\end{pmatrix}$ $=-\begin{vmatrix}1 & 1 & 2\\ 0 & -1 & 0\\ 0 & 0 & -3\end{vmatrix}$ $(R3:=R3-5R2)$

$=-(\text{product of diagonal entries})=-(1)(-1)(-3)=\boxed{-3}$

(b) Expanding along the first row we get

$\det(B)=2\cdot(-1)^{1+1}\cdot\begin{vmatrix}1 & 2\\ -2 & 3\end{vmatrix}+1\cdot(-1)^{1+2}\cdot\begin{vmatrix}1 & 2\\ 3 & 3\end{vmatrix}+4\cdot(-1)^{1+3}\cdot\begin{vmatrix}1 & 1\\ 3 & -2\end{vmatrix}$

$=2(3-(-4))+(-1)\cdot(3-6)+4(-2-3)$

$=14+3-20=\boxed{-3}$

4(a) The set $\{v_1, v_2, \dots, v_k\}$ of vectors is linearly independent if $c_1\underset{\sim}{v_1}+c_2\underset{\sim}{v_2}+\dots+c_k\underset{\sim}{v_k}=\underset{\sim}{0}\Rightarrow c_1=c_2=\dots=c_k=0$.

(b) Suppose $c_1\begin{pmatrix}-1\\ 1\\ 3\end{pmatrix}+c_2\begin{pmatrix}-2\\ 1\\ 1\end{pmatrix}+c_3\begin{pmatrix}-4\\ 1\\ -3\end{pmatrix}=\underset{\sim}{0}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$. Then

$\left.\begin{aligned}-c_1-2c_2-4c_3&=0\\ c_1+c_2+c_3&=0\\ 3c_1+c_2-3c_3&=0\end{aligned}\right\} \rightarrow \left.\begin{aligned}c_1+2c_2+4c_3&=0\\ -c_2-3c_3&=0\\ -5c_2-15c_3&=0\end{aligned}\right\}$ $\begin{aligned}&E1:=(-1)E1\\ &E2:=E2+E1\\ &E3:=E3+3E1\end{aligned}$

4(b) $\rightarrow$ $\left.\begin{aligned} c_1 \quad - 2c_3 &= 0 \\ c_2 + 3c_3 &= 0 \\ 0 &= 0 \end{aligned}\right\}$ $\begin{aligned} &E1 := E1 + 2E2 \\ &E2 := (-1)E2 \\ &E3 := E3 - 5E2 \end{aligned}$ Take $c_3 = 1$. Then $c_1 = 2$ and $c_2 = -3$.

So $2\underset{\sim}{v}_1 + (-3)\underset{\sim}{v}_2 + \underset{\sim}{c}_3 = \underset{\sim}{0}$. $\therefore$ $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + c_3\underset{\sim}{v}_3 = \underset{\sim}{0} \not\Rightarrow$ $c_1 = c_2 = c_3 = 0$. Hence $\{\underset{\sim}{v}_1, v_2, v_3\}$ is not linearly independent.

5(a) A set, S, of vectors in $\mathbb{R}^n$ is a subspace if $S \neq \emptyset$, $\underset{\sim}{u} + \underset{\sim}{v} \in S$ whenever $\underset{\sim}{u}, \underset{\sim}{v} \in S$, and $\alpha\underset{\sim}{u} \in S$ whenever $\underset{\sim}{u} \in S$ & $\alpha \in \mathbb{R}$.

(b) $$\{\underset{m\times n}{(A+B)}\underset{n\times p}{C}\}[i,j] = \sum_{k=1}^{n}(A+B)[i,k].C[k,j] = \sum_{k=1}^{n}\{A[i,k] + B[i,k]\}.C[k,j]$$
$$= \sum_{k=1}^{n}\{A[i,k].C[k,j] + B[i,k].C[k,j]\}$$
$$= \sum_{k=1}^{n} A[i,k].C[k,j] + \sum_{k=1}^{n} B[i,k].C[k,j]$$
$$= (AC)[i,j] + (BC)[i,j] = (AC+BC)[i,j].$$
$\therefore$ $(A+B)C = (AC) + (BC)$.

6 (a) M_{ij} = the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j from A. $A_{ij} = (-1)^{i+j}.\det(M_{ij})$

(b) <u>Basis</u>: If $n=2$, then $\det(A') = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$

$= -(a_{11}a_{22} - a_{21}a_{12}) = -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -\det(A)$.

So result is true for $n=2$.

<u>Ind. Step:</u> Suppose the result is true for all $(n-1)\times(n-1)$ matrices where $n \geq 3$. Then by expanding $\det(A')$ along row k, with $k \neq i$ or j, we get $\det(A') = a_{k1}A'_{k1} + a_{k2}.A'_{k2} + \dots + a_{kn}A'_{kn}$

$a_{k1}.(-1)^{k+1}.\det[M_{k1}(A')] + a_{k2}.(-1)^{k+2}.\det[M_{k2}(A')] + \dots + (-1)^{k+n}.[\det M_{kn}(A')]$

$= a_{k1}.(-1)^{k+1}.(-\det[M_{k1}(A)]) + a_{k2}.(-1)^{k+2}.(-\det[M_{k2}(A)]) + \dots + (-1)^{k+n}.(-\det[M_{kn}(A)])$

$= -\{a_{k1}A_{k1} + a_{k2}A_{k2} + \dots + a_{kn}A_{kn}\} = -\det(A)$. So if the result is true for $(n-1)\times(n-1)$ matrices, it will be true for $n\times n$ matrices.

<u>Conclusion</u>: Hence by the Principle of Mathematical Induction, $\det(A') = -\det(A)$ for all $n\times n$ matrices with $n \geq 2$.