MAS 3305 - LINEAR ALGEBRA — FLORIDA INT'L UNIV.

TEST #2 - SPRING 2016 — TIME: 75 min.

Answer all 6 questions. NO CALCULATORS or CELL PHONES ARE ALLOWED. Show all working and provide all reasoning where required. An unjustified answer will receive little or no credit. **Begin each of the 6 questions on 6 separate pages.**

Let $E = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle$, $G = \left\langle \begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}4\\-7\end{pmatrix} \right\rangle$, and $H = \left\langle \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \right\rangle$.

(15) 1.(a) Find the *transition matrix*, ${}_E(I)_G$, from the basis G to the basis E; and the *transition matrix*, ${}_H(I)_E$, from E to H.

(b) Find the *transition matrix*, ${}_H(I)_G$, from the basis G to the basis H; and the *transition matrix*, ${}_G(I)_H$, from H to G.

Let $A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ -1 & 2 & 1 & 5 \\ 2 & -4 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 2 \\ 5 & 2 & 5 \\ 3 & 1 & 1 \end{pmatrix}$.

(25) 2.(a) Find bases for the *row space* and the *null space* of the matrix A.

(b) Find bases for the *co-null space* and the *column space* of the matrix B.

(15) 3.(a) Find the *best least squares fit* by *a linear function* $c_0 + c_1.x$ to the data below.

(b) *Check how close* your new answers are to the given values of $y(x)$.

x	0	2	3
$y(x)$	2	1	0

x	0	2	3
$c_0 + c_1.x$	?	?	?

(15) 4. Using the *Gram-Schmidt orthogonalization process*, find an *orthonormal basis* of the subspace $S = \text{span}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix} \right\}$ and check it is orthonormal.

(15) 5.(a) Let L be a linear transformation from the V to W. If R is a subset of V and S is a subset of W, define what are L[R] and L^{-1}[S].

(b) If L is a linear transformation from V to W and R is a *subspace* of V, prove that L[R] is a *subspace* of W.

(15) 6.(a) Let A be an mxn matrix. Define what are *RowSp(A)* and *ColSp(A)*.

(b) Let $\{(\underline{u}_1)^T, (\underline{u}_2)^T, \ldots, (\underline{u}_k)^T\}$ be a basis of *RowSp(A)* and put $\underline{v}_i = A\underline{u}_i$. Prove that $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is a linearly independent set of vectors in *ColSp(A)*.

1(a) $_E(I)_G = \begin{bmatrix}1 & 4\\ -2 & -7\end{bmatrix}$, $\quad {}_H(I)_E = \{{}_E(I)_H\}^{-1} = \begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}^{-1} = \begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}$

(b) $_H(I)_G = {}_H(I)_E\,{}_E(I)_G = \begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\ -2 & -7\end{bmatrix} = \begin{bmatrix}5 & 19\\ -4 & -15\end{bmatrix}$

$_G(I)_H = {}_G(I)_E\,{}_E(I)_H = \begin{bmatrix}1 & 4\\ -2 & -7\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} = \begin{bmatrix}-7 & -4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} = \begin{bmatrix}-15 & -19\\ 4 & 5\end{bmatrix}$

Check: $_G(I)_H = \{{}_H(I)_G\}^{-1} = \begin{bmatrix}5 & 19\\ -4 & -15\end{bmatrix}^{-1} = \frac{1}{(-75)+76}\begin{bmatrix}-15 & -19\\ 4 & 5\end{bmatrix} = \begin{bmatrix}-15 & -19\\ 4 & 5\end{bmatrix}$ ✓

2(a) $\begin{bmatrix}1 & -2 & 1 & -1\\ -1 & 2 & 1 & 5\\ 2 & -4 & 3 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & -2 & 1 & -1\\ 0 & 0 & 2 & 4\\ 0 & 0 & 1 & 2\end{bmatrix}$ $\quad R2 := R2 + R1$, $\; R3 := R3 - 2R1$

$\longrightarrow \begin{bmatrix}1 & -2 & 0 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2\end{bmatrix}$ $\quad R1 := R1 - R3$, $\; R2 := R2 - 2R3$ $\quad \longrightarrow \begin{bmatrix}1 & -2 & 0 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = A_s$ ← insert row of zeros

$I_4 - A_s = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}1 & -2 & 0 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 2 & 0 & 3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}$

Basis of RowSp(A) $= \{(1,-2,0,-3),\ (0,0,1,2)\}$
Basis of NullSp(A) $= \{(2,1,0,0)^T,\ (3,0,-2,1)^T\}$

{ Check answer by verifying that these two sets are orthogonal to each other.

(b) $\left[\begin{array}{ccc|cccc}1 & 1 & -1 & 1 & 0 & 0 & 0\\ -2 & 0 & 2 & 0 & 1 & 0 & 0\\ 5 & 2 & 5 & 0 & 0 & 1 & 0\\ 3 & 1 & 1 & 0 & 0 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|cccc}1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 2 & 1 & 0 & 0\\ 0 & -3 & 10 & -5 & 0 & 1 & 0\\ 0 & -2 & 4 & -3 & 0 & 0 & 1\end{array}\right]$ $\quad R2 := R2 + 2R1$, $\; R3 := R3 - 5R1$, $\; R4 := R4 - 3R1$

$\longrightarrow \left[\begin{array}{ccc|cccc}1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 2 & 1 & 0 & 0\\ 0 & -6 & 20 & -10 & 0 & 2 & 0\\ 0 & 0 & 4 & -1 & 1 & 0 & 1\end{array}\right]$ $\quad R3 := 2R3$, $\; R4 := R4 + 2R2$ $\quad \longrightarrow \left[\begin{array}{ccc|cccc}1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 2 & 1 & 0 & 0\\ 0 & 0 & 20 & -4 & 3 & 2 & 0\\ 0 & 0 & 20 & -5 & 5 & 0 & 5\end{array}\right]$ $\quad R3 := R3 - 3R2$, $\; R4 := 5R4$

$\longrightarrow$ Row Echelon Form: $\left[\begin{array}{ccc|cccc}1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 2 & 1 & 0 & 0\\ 0 & 0 & 20 & -4 & 3 & 2 & 0\\ 0 & 0 & 0 & -1 & 2 & -2 & 5\end{array}\right]$ $\quad R4 := R4 - R3$

2(b) Basis of $ColSp(A) = \{(1,2,5,3)^T, (1,0,2,1)^T, (-1,2,5,1)^T\}$

Basis of $CoNullSp(A) = \{(-1,2,-2,5)\}$. (Check answers by verifying that these two sets are orthogonal to each other.)

3(a) Suppose $y(x) = c_0 + c_1 x$. Then

$$\begin{aligned} c_0 + c_1 \cdot 0 &= y(0) = 2 \\ c_0 + c_1 \cdot 2 &= y(2) = 1 \\ c_0 + c_1 \cdot 3 &= y(3) = 0 \end{aligned}$$

$$\therefore \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{b}}.$$

Normal system is $(A^T A)\hat{\underset{\sim}{c}} = \underset{\sim}{b}$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 5 \\ 5 & 13 \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{39-25} \begin{bmatrix} 13 & -5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 29 \\ -9 \end{bmatrix} = \begin{pmatrix} 29/14 \\ -9/14 \end{pmatrix}.$$

So $\hat{c}_0 = 29/14$ & $\hat{c}_1 = -9/14$.

3(b) Check:

x	0	2	3
$y(x)$	2	1	0
$\hat{c}_0 + \hat{c}_1 x$	29/14	11/14	2/14

4(a) $\underset{\sim}{v}_1 = \underset{\sim}{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\quad \underset{\sim}{v}_2 = \underset{\sim}{x}_2 - \left(\frac{\underset{\sim}{x}_2 \cdot \underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|^2} \right) \underset{\sim}{v}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$

$$\underset{\sim}{v}_3 = \underset{\sim}{x}_3 - \left(\frac{\underset{\sim}{x}_3 \cdot \underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|^2} \right) \underset{\sim}{v}_1 - \left(\frac{\underset{\sim}{x}_3 \cdot \underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|^2} \right) \underset{\sim}{v}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{12}{12} \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

$$\therefore \hat{\underset{\sim}{u}}_1 = \frac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{\underset{\sim}{u}}_2 = \frac{\underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \hat{\underset{\sim}{u}}_3 = \frac{\underset{\sim}{v}_3}{\|\underset{\sim}{v}_3\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

(b) Check: $\hat{\underset{\sim}{u}}_1 \cdot \hat{\underset{\sim}{u}}_2 = \frac{1}{\sqrt{6}}(1-1) = 0$, $\hat{\underset{\sim}{u}}_1 \cdot \hat{\underset{\sim}{u}}_3 = \frac{1}{\sqrt{12}}(-1+1) = 0$, $\hat{\underset{\sim}{u}}_2 \cdot \hat{\underset{\sim}{u}}_3 = \frac{1}{\sqrt{18}}(-1+2-1) = 0$.

5(a) $L[R] = \{L(\underset{\sim}{x}) : \underset{\sim}{x} \in R\}$, $\quad L^{-1}[S] = \{\underset{\sim}{x} \in V : L(\underset{\sim}{x}) \in S\}$.

(b) Suppose $L: V \to W$ is a linear transformation and R is a subspace of V. Then $\underset{\sim}{0}_V \in R$ because R is a subspace of V. So $L(\underset{\sim}{0}_V) \in L[R]$ and hence $L[R] \neq \emptyset$. Now let $\underset{\sim}{y}_1, \underset{\sim}{y}_2 \in L[R]$ and $\alpha \in \mathbb{R}$. Then we can find $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in R$ such that $L(\underset{\sim}{x}_1) = \underset{\sim}{y}_1$ and $L(\underset{\sim}{x}_2) = \underset{\sim}{y}_2$. So $\underset{\sim}{y}_1 + \underset{\sim}{y}_2 = L(\underset{\sim}{x}_1) + L(\underset{\sim}{x}_2) \overset{*}{=} L(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) \in L[R]$ and $\alpha \underset{\sim}{y}_1 = \alpha L(\underset{\sim}{x}_1) \overset{*}{=} L(\alpha \underset{\sim}{x}_1) \in L[R]$. $\therefore L[R]$ is a subspace of W.

(*) is true bec. L is a linear map.

6 (a) $RowSp(A) = \{\vec{x}A : \vec{x} \in \vec{\mathbb{R}}^m\}$ $\quad ColSp(A) = \{A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n\}$.

(b) Suppose $c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \cdots + c_k \underset{\sim}{v_k} = \underset{\sim}{0}$. Then

$$
\begin{aligned}
\underset{\sim}{0} &= c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \cdots + c_k \underset{\sim}{v_k} \\
&= c_1(A\underset{\sim}{u_1}) + c_2(A\underset{\sim}{u_2}) + \cdots + c_k(A\underset{\sim}{u_k}) \\
&= A(c_1\underset{\sim}{u_1}) + A(c_2\underset{\sim}{u_2}) + \cdots + A(c_k\underset{\sim}{u_k}) \\
&= A(c_1\underset{\sim}{u_1} + c_2\underset{\sim}{u_2} + \cdots + c_k\underset{\sim}{u_k}) = A\underset{\sim}{x}, \text{ say}
\end{aligned}
$$

where $\underset{\sim}{x} = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$. So $\vec{r}_i \underset{\sim}{x} = 0$ for each row $\vec{r}_i$ of A $(i = 1, \ldots, m)$. But $\underset{\sim}{x}^T$ is a linear combination of $\vec{r}_1, \vec{r}_2, \ldots, \vec{v}_m$ because $\{\underset{\sim}{u_1}^T, \ldots, \underset{\sim}{u_k}^T\}$ is a basis of $RowSp(A)$. Hence $\underset{\sim}{x}^T \underset{\sim}{x} = 0$. Thus $\|\underset{\sim}{x}^T\|^2 = 0$ which implies $\underset{\sim}{x}^T = \vec{0}$. Thus $\vec{0} = \underset{\sim}{x}^T = (c_1 u_1 + c_2 u_2 + \cdots + c_k \underset{\sim}{u_k})^T = c_1 \underset{\sim}{u_1}^T + c_2 \underset{\sim}{u_2}^T + \cdots + c_k u_k^T$. Since $\{\underset{\sim}{u_1}^T, \ldots, u_k^T\}$ is a basis, we must have $c_1 = c_2 = \cdots = c_k = 0$. Hence $\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k}\}$ is linearly independent.

END