Notes for section 4.3 :

Theorem 5 : Let $L_2 : \mathbb{R}^p \to \mathbb{R}^n$ & $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Then

(a) $L_1 \circ L_2 : \mathbb{R}^p \to \mathbb{R}^m$ is also a linear transformation, and

(b) $${}_{E_m}(L_1 \circ L_2)_{E_p} = {}_{E_m}(L_1)_{E_n} \; {}_{E_n}(L_2)_{E_p}$$

Proof: (b) Let $A = [a_{ij}] = {}_{E_m}(L_1 \circ L_2)_{E_p}$, $B = [b_{ij}] = {}_{E_m}(L_1)_{E_n}$ and $C = [c_{ij}] = {}_{E_n}(L_2)_{E_p}$. Then for any $j = 1, \dots, p$

$$\text{column } j \text{ of } A = (L_1 \circ L_2)(\underset{\sim}{e}_j) = L_1(L_2(\underset{\sim}{e}_j))$$

$$= L_1(c_{1j}\underset{\sim}{e}_1 + c_{2j}\underset{\sim}{e}_2 + \cdots + c_{nj}\underset{\sim}{e}_n)$$

$$= c_{1j}L_1(\underset{\sim}{e}_1) + c_{2j}L_1(\underset{\sim}{e}_2) + \cdots + c_{nj}L_1(\underset{\sim}{e}_n)$$

$$\begin{aligned} = \; & c_{1j}(b_{11}\underset{\sim}{e}_1 + b_{21}\underset{\sim}{e}_2 + \cdots + b_{m1}\underset{\sim}{e}_m) \\ + \; & c_{2j}(b_{12}\underset{\sim}{e}_1 + b_{22}\underset{\sim}{e}_2 + \cdots + b_{m2}\underset{\sim}{e}_m) \\ & \vdots \\ + \; & c_{nj}(b_{1n}\underset{\sim}{e}_1 + b_{2n}\underset{\sim}{e}_2 + \cdots + b_{mn}\underset{\sim}{e}_m) \end{aligned}$$

$$\begin{aligned} = \; & (b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1n}c_{nj})\underset{\sim}{e}_1 \\ + \; & (b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2n}c_{nj})\underset{\sim}{e}_2 \\ & \vdots \\ + \; & (b_{m1}c_{1j} + b_{m2}c_{2j} + \cdots + b_{mn}c_{nj})\underset{\sim}{e}_m \end{aligned}$$

$$= \begin{bmatrix} (\text{row } 1 \text{ of } B).(\text{column } j \text{ of } C) \\ (\text{row } 2 \text{ of } B).(\text{column } j \text{ of } C) \\ \vdots \\ (\text{row } m \text{ of } B.(\text{column } j \text{ of } C) \end{bmatrix} = \text{column } j \text{ of } (BC)$$

$\therefore$ column $j$ of $A$ = column $j$ of $(BC)$ for $j = 1, \dots, p$.

$\therefore$ $A = BC$. Hence ${}_{E_m}(L_1 \circ L_2)_{E_p} = {}_{E_m}(L_1)_{E_n} \; {}_{E_n}(L_2)_{E_p}$.

**Def.** The $m \times n$ matrix $B$ is equivalent to the $m \times n$ matrix $A$ if we can find invertible matrices $P$ and $Q$ such that $B = Q^{-1}AP$

**Fact 1**: If $B$ is equivalent to $A$, then $A$ is equivalent to $B$.

**Proof.** Suppose $B$ is equivalent to $A$. Then we can find invertible matrices $P$ & $Q$ such that $B = Q^{-1}AP$
So $QBP^{-1} = QQ^{-1}APP^{-1} = IAI = A$. Thus

$$A = QBP^{-1} = (Q^{-1})^{-1}B(P^{-1})$$

and since $Q^{-1}$ & $P^{-1}$ are invertible, it follows that $A$ is similar to $B$.

**Def.** The $n \times n$ matrix $B$ is similar to the $n \times n$ matrix $A$ if we can find an invertible matrix $P$ such that $B = P^{-1}AP$.

**Fact 2**: If $B$ is similar to $A$, then $A$ is similar to $B$.

**Proof.** Suppose $B$ is similar to $A$. Then we can find an invertible matrix $P$ such that $B = P^{-1}AP$. So

$$PBP^{-1} = PP^{-1}APP^{-1} = IAI = A$$

$\therefore \quad A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$. Since $P^{-1}$ is an invertible matrix it follows that $A$ is similar to $B$.

**Fact 3**: (a) Two $m \times n$ matrices $A$ & $B$ represent the same linear transf. $L: \mathbb{R}^n \to \mathbb{R}^m \Leftrightarrow A$ & $B$ are equivalent.
(b) Two $n \times n$ matrices $A$ & $B$ represent the same lin. transf $L: \mathbb{R}^n \to \mathbb{R}^n$ w.r.t. the same basis $\Leftrightarrow A$ & $B$ are similar.