Supplementary Problems - Sec. 3.6

Linear Algebra
MAS 3105 - SP'2014

1. Let $A = \begin{bmatrix} 0 & 0 & 2 & 6 \\ 1 & -1 & 2 & 4 \\ 1 & -1 & 1 & 1 \end{bmatrix}$

(a) Find a basis C for the column space of A and a basis B for the co-null space of A.

(b) Check that each vector in $[B]^T$ is orthogonal to each vector in [C]. (Here $[B]^T$ is the set of vectors which are the transposes of the row vectors in B.)

(c) Find a basis D for the row space of A and a basis E for the nullspace of A.

(d) Check that each vector in $[D]^T$ is orthogonal to each vector in [E].

2. Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 1 & -2 & 3 \\ 1 & -2 & 5 \end{bmatrix}$

(a) Find a basis for the column space of A and a basis for the co-null space of A.

(b) If [B] is the matrix with the basis of CoNull(A) as its rows and [C] is the matrix with the basis of ColSp(A) as its columns, show that [B][C] = O.

(c) Find a basis for the row space of A and a basis of the nullspace of A.

(d) If [D] is the matrix with the basis of RowSp(A) as its rows and [E] is the matrix with the basis of Null(A) as its columns, show that [D][E] = O.

1(a)

$$\left[\begin{array}{cccc|ccc} \underset{\uparrow}{0} & 0 & \underset{\uparrow}{2} & 6 & 1 & 0 & 0 \\ 1 & -1 & 2 & 4 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 6 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R1:=R3 \\ \\ R3:=R1 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 6 & 1 & 0 & 0 \end{array}\right] R2:=R2-R1$$

$$\rightarrow U = \left[\begin{array}{cccc|ccc} \underline{1} & -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & \underline{1} & 3 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right] R3:=R3-2R2$$

So a basis for the column space of A will consist of the 1st & 3rd columns of A. Thus $C=\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$.

Also a basis for the co-null space of A consists of the rows to the right of the rows of zeros in the row echelon form U of A. Thus $B=\{(1,-2,2)\}$.

(b) $[B]^T=\left\{\begin{pmatrix}1\\-2\\2\end{pmatrix}\right\}$, $C=\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$.

$$\begin{pmatrix}1\\-2\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0-2+2=0, \quad \begin{pmatrix}1\\-2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\2\\1\end{pmatrix}=2-4+2=0. \checkmark$$

(c)

$$\begin{bmatrix} 0 & 0 & 2 & 6 \\ 1 & -1 & 2 & 4 \\ 1 & -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 2 & 4 \\ 0 & 0 & 2 & 6 \end{bmatrix} \begin{array}{l} R1:=R3 \\ \\ R3:=R1 \end{array}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix} R2:=R2-R1$$

$$\begin{array}{r} \rightarrow \\ A_R = \\ \text{delete} \rightarrow \end{array} \begin{bmatrix} \underline{1} & -1 & 0 & -2 \\ 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R1:=R1-R2 \\ \\ R3:=R3-2R2 \end{array}$$

1(c)

$$A_S = \begin{array}{ccccc} & & \downarrow & & \downarrow \\ \rightarrow & \underline{1} & -1 & 0 & -2 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & \underline{1} & 3 \\ & 0 & 0 & 0 & 0 \end{array}$$

new row of 0's

new row of 0's

To find a basis for the row space of A, we just take the non-zero rows of the reduced row row echelon form A_R of A. So

$$D = \{(1,-1,0,-2),\ (0,0,1,3)\}$$

To find a basis for the null space of A, we add or delete rows of zeros to get a square matrix (called the supplemented matrix A_s) with the leading in the diagonal position. The nonzero columns of $(I-A_s)$ will give us the basis E.

$$I-A_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{So} \quad E = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

(d)

$$[D]^T = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}, \quad E = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1-1+0+0=0, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} = 2+0+0+2=0.$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0+0+0+0=0, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} = 0+0-3+3=0.$$

<u>Note</u>: You can also find a basis for Null(A) by solving the system of equations $[A_R]\,\underline{x} = \underline{0}$.

2(a)

$$\left[\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 2 & -4 & 4 & 0 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 & 0 \\ 1 & -2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2 := R2-2R1 \\ R3 := R3-R1 \\ R4 := R4-R1 \end{array}$$

$\uparrow \qquad \uparrow$

$$\to \left[\begin{array}{ccc|cccc} 1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2 := R3 \\ R3 := R2 \\ \\ \end{array}$$

$$\to U = \left[\begin{array}{ccc|cccc} \underline{1} & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -3 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R3 := R3-3R2 \end{array}$$

Since the leading 1's of the row echelon form U of A are in columns 1 & 3, a basis for the column space of A consists of columns 1 & 3 of A. Thus $C = \left\{ \begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}2\\4\\3\\5\end{pmatrix} \right\}$. Also a basis for the co-null space of A consist of the rows to the right of the rows of zeros of U. Thus

$$B = \{(-2,1,0,0), (2,0,-3,1)\}$$

(b) $[B] = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$

So

$$[B][C] = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2,2}$$

2(c)

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 1 & -2 & 3 \\ 1 & -2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{matrix} R2 := R2 - 2R1 \\ R3 := R3 - R1 \\ R4 := R4 - R1 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{matrix} R2 := R3 \\ R3 := R2 \end{matrix}$$

$$\longrightarrow A_R \begin{bmatrix} \underline{1} & -2 & 0 \\ 0 & 0 & \underline{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R1 := R1 - 2R2$$

$\leftarrow \}$ delete these rows

$$\longrightarrow A_S \begin{array}{c} \begin{bmatrix} \underline{1} & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \underline{1} \end{bmatrix} \\ \uparrow \end{array}$$

$\leftarrow$ and then add a new row of zero's to get A_S.

To find a basis for the row space of A, we just take the non-zero rows of the reduced row echelon form A_R of A. So $D = \{ (1,-2,0), (0,0,1) \}$.

To find a basis for the nullspace of A, we add or delete rows of zeros from the matrix A_R to get a square matrix A_S with the leading 1's on the diagonal position. The non-zero columns of $I - A_S$ will give us the basis E. $I - A_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \uparrow \end{array}$.

Thus $E = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

(d) $[D] = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. So

$$[D][E] = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O_{2,1}.$$