A1

MHF 4102 - AX. SET THEORY — FLORIDA INT'L UNIV.

PROBLEMS FOR CHAPTER 1 — SPRING 2004

1. Prove the following directly from the definitions
   (a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
   (b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Y)$
   (c) $X - Y = X - (X \cap Y)$

2. If X and Y are both subsets of Z prove that
   (a) $Z - (X \cup Y) = (Z - X) \cap (Z - Y)$
   (b) $Z - (X \cap Y) = (Z - X) \cup (Z - Y)$
   (c) $Z - (Z - X) = X$

3. Find
   (a) $\{\emptyset\} - \emptyset$
   (b) $\{\emptyset, \{\emptyset\}\} - \{\{\emptyset\}\}$
   (c) $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\}$
   (d) $\{\{\emptyset\}\} - \{\emptyset\}$
   (e) $\{\{\emptyset\}\} \cup \{\emptyset, \{\emptyset\}\}$
   (f) $\{\emptyset\} - \{\{\emptyset\}\}$

4. If x and Y are sets, what are
   (a) $\bigcup \{X\}$
   (b) $\bigcap \{X\}$
   (c) $\bigcup \{X, Y\}$
   (d) $\bigcap \{X, Y\}$
   (e) $\bigcup \emptyset$
   (f) $\bigcap \emptyset$?

5. Let $\langle X_i : i \in I \rangle$ be a family of subsets of Z. Prove that
   (a) $Z - \bigcup_{i \in I} X_i = \bigcap_{i \in I} (Z - X_i)$
   (b) $Z - \bigcap_{i \in I} X_i = \bigcup_{i \in I} (Z - X_i)$

6. Let R and S be the binary relations on $\mathbb{N}$ defined by
   aRb if a is a multiple of b
   aSb if a and b has no common factor.

   Determine whether or not R and S are
   (a) reflexive
   (b) symmetric
   (c) transitive
   (d) connected
   (e) anti-symmetric.

7. Let $A = \{1,2\}$. Enumerate all the binary relations on A. (Hint: There are 16 of them)

8. Let $A = \{a, b\}$ and $B = \{1,2,3\}$.
   (a) How many functions are there from A to B ?
   (b) How many of these functions are injections ?

9. (a) How many functions are there from $\{1,2\}$ to $\emptyset$
   (b) " " " " " $\emptyset$ to $\{1,2\}$
   (c) " " " " " $\emptyset$ to $\emptyset$

10. Let $f: X \to Y$ be a function, and A and B be subsets of Y. Prove that
    (a) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$
    (b) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$
    (c) $f^{-1}[A - B] = f^{-1}[A] - f^{-1}[B]$.

11. Let $f: X \to Y$ be a function and $A_i \subseteq Y$ for each $i \in I$. Which of the following are true
    (a) $f^{-1}[\bigcup_{i \in I} A_i] = \bigcup_{i \in I} f^{-1}[A_i]$
    (b) $f^{-1}[\bigcap_{i \in I} A_i] = \bigcap_{i \in I} f^{-1}[A_i]$?

12. Suppose $f: A \to B$ and $g: B \to C$ are both injective functions. Does it follow that $g \circ f: A \to C$ must also be an injective function.

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PROBLEMS FOR CHAPTER 2 — FLORIDA INT'L UNIV.

1. (a) Write down all the elements of V_4 in the cumulative hierarchy of sets.
   (b) Find a formula for the number of elements of V_n in terms of n only.

2. Write out each of the 10 axioms completely in the language of set theory. For example

   Nullset Axiom: $(\exists x_1)(\forall x_2)(\neg(x_2 \in x_1))$

   Extensionality Axiom:

   $$(\forall x_1)(\forall x_2)\Big(\big((\forall x_3)(x_3 \in x_1 \rightarrow x_3 \in x_2)\ \&\ (\forall x_4)(x_4 \in x_2 \rightarrow x_4 \in x_1)\big) \rightarrow (x_1 = x_2)\Big)$$

3. Let V = collection of all sets. Prove that V is a proper class.
   (Hint: Use the separation axiom and the fact that $R = \{x : x \notin x\}$ is a proper class.)

4. Let $\mathcal{A}$ be a class.
   (a) If $\mathcal{A}$ is a set prove that $\mathcal{A}^c$ is a proper class.
   (b) If $\mathcal{A}$ is a proper class does it follow that $\mathcal{A}^c$ is a set?

5. Prove that for any set A, $\mathcal{P}(A) \notin A$.
   (Hint: Let $D = \{a \in A : a \notin a\}$. Show that $D \in \mathcal{P}(A)$, but $D \notin A$.)

6. Use the Foundation axiom to show that there is no set x such that $x \in x$.
(Hint: Suppose x was a set such that $x \in x$. Let $A = \{x\}$. Show that A has no element a such that $a \cap A = \emptyset$.)

7. Let $\langle A, \leq \rangle$ be an ordered set. An element a is said to be a maximal element of A if there is no x in A with $a < x$.
Find the smallest ordered sets with
(a) 5 maximal elements and 3 minimal elements.
(b) 2 maximal elements and 4 minimal elements.

8. Let $\langle A, \leq \rangle$ be an ordered set such that any non-empty subset of A has a smallest element. Prove that "$\leq$" must be a linear ordering on A.

9. Let A be a non-empty set and R be the relation on A defined by
aRb if $a \in b$.
(a) Is it possible for R to be transitive?
(b) Is R always irreflexive ?
(c) Is it possible for R to be symmetric?
(d) Is it possible for R to be connected?
(e) Is R always asymmetric ?
Def. R is irreflexive if $a \not R a$ for all $a \in A$.

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PROBLEMS FOR CHAPTER 3 SPRING 2004

1. Find a linearly ordered set $\langle L, < \rangle$ and an initial segment S of L such that S is not of the form $\{x : x < a\}$ with $a \in L$.

2. Find a linearly ordered set $\langle L, < \rangle$ and an increasing function $f : L \to L$ such that $f(x) > x$ for at least one $x \in L$ and $f(x) < x$ for at least one $x \in L$.

3. Explain why $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is not an ordinal

4. Prove that A is transitive if and only if $a \in A \Rightarrow a \subseteq A$.

5. Determine which of the following statements are true and which are not.
   (a) If X and Y are transitive, so is $X \cap Y$
   (b) If X and Y are transitive, so is $X \cup Y$
   (c) If $X \in Y$ and Y is transitive, then X is transitive
   (d) If $X \subseteq Y$ and Y is transitive, then X is transitive
   (e) If every element in X is transitive, then X is transitive.

6. Prove that
   (a) If $\mathcal{A}$ is a set of ordinals, $\bigcup \mathcal{A}$ is an ordinal
   (b) If $\mathcal{A}$ is a non-empty set of ordinals, then $\bigcap \mathcal{A}$ is an ordinal.

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7. Recall that V_ω was defined inductively as follows:

$$V_0 = \emptyset$$
$$V_{n+1} = \mathcal{P}(V_n) \quad \text{for each } n \in \omega$$
$$V_\omega = \bigcup_{n \in \omega} V_n$$

Prove that

(a) V_ω is transitive

(b) $x \in V_\omega \Rightarrow \{x\} \in V_\omega$

(c) $x \in V_\omega \Rightarrow x \cup \{x\} \in V_\omega$.

8. (a) Prove the associative law for ordinal mult.

$$(\alpha . \beta) . \gamma = \alpha . (\beta . \gamma)$$

(b) Prove the following distributive law for ordinals

$$\alpha . (\beta + \gamma) = (\alpha . \beta) + (\alpha . \gamma)$$

9. Show that the following statements are not always true.

(a) If $\alpha + \gamma = \beta + \gamma$, then $\alpha = \beta$

(b) If $\gamma > 0$ and $\alpha . \gamma = \beta . \gamma$, then $\alpha = \beta$

(c) $(\beta + \gamma) . \alpha = (\beta . \alpha) + (\gamma . \alpha)$

10. Simplify:

(a) $(\omega + 2) + \omega$

(b) $\omega + \omega^2$

(c) $(\omega + 2) . (\omega + 3)$

(d) $(\omega + 1)^2 . \omega^3$

11. Prove that an ordinal α is a limit ordinal $\Leftrightarrow$ $\alpha = \omega . \beta$ for some ordinal β.

12. Find the smallest ordinal $\alpha \neq 0$ such that

(a) $\omega + \alpha = \alpha$ (b) $\omega . \alpha = \alpha$ (c) $\omega^\alpha = \alpha$.

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PROBLEMS FOR CHAPTER 4 — SPRING 2004

1. Let $F(A,B)$ = set of all functions from A to B. Prove that for any sets A, B, & C
   (a) $A \preccurlyeq F(A,A)$
   (b) $A \subseteq B \Rightarrow F(A,C) \preccurlyeq F(B,C)$
   (c) $|B| \geq 2 \Rightarrow A \prec F(A,B)$
   (d) $B \subseteq C \Rightarrow F(A,B) \preccurlyeq F(A,C)$

2. Prove that
   (a) $\mathcal{P}(\mathbb{N}) \approx F(\mathbb{N},2)$ (Remember $2 = \{0,1\}$.)
   (b) $F(\mathbb{N},\mathbb{N}) \approx F(\mathbb{N},2)$

3. Prove that
   (a) $\mathbb{A} \approx \mathbb{N}$ ($\mathbb{A}$ = set of alg. nos.)
   (b) $\mathbb{R} \approx [0,1)$
   (c) $\mathbb{R} \approx F(\mathbb{N},2)$ <u>Hint</u>: Show that $[0,1) \approx F(\mathbb{N},2)$

4. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be an infinite sequence of 0's and 1's. We say that $\langle a_n \rangle$ has <u>finite support</u> if the number of 1's in $\langle a_n \rangle$ is finite. Prove that
   (a) the set of all sequences of 0's and 1's with finite support is equipotent to $\mathbb{N}$.
   (b) the set of all <u>finite</u> subsets of $\mathbb{N}$ is equipotent to $\mathbb{N}$.

5. Let $B(\mathbb{N},\mathbb{N})$ = set of all bijections from $\mathbb{N}$ to $\mathbb{N}$. Is $B(\mathbb{N},\mathbb{N}) \approx F(\mathbb{N},\mathbb{N})$?

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PROBLEMS FOR CHAPTER 4 SPRING 2004

6. For any cardinal numbers κ, μ, ν prove that
   (a) $\kappa+\mu = \mu+\kappa$
   (b) $(\kappa+\mu)+\nu = \kappa+(\mu+\nu)$
   (c) $\kappa+\kappa = 2.\kappa$

7. For any cardinals numbers κ, μ, ν prove that
   (a) $\kappa.\mu = \mu.\kappa$
   (b) $(\kappa.\mu).\nu = \kappa.(\mu.\nu)$
   (c) $\kappa.(\mu+\nu) = \kappa.\mu + \kappa.\nu$

8. For any cardinal number κ, μ, ν prove that
   (a) $\kappa.\kappa = \kappa^2$
   (b) $\kappa^{\mu+\nu} = \kappa^{\mu}.\kappa^{\nu}$
   (c) $(\kappa.\mu)^{\nu} = \kappa^{\nu}.\mu^{\nu}$
   (d) $(\kappa^{\mu})^{\nu} = \kappa^{\mu.\nu}$

9. Determine which of the following are true
   (a) $\mu < \nu \Rightarrow \kappa+\mu < \kappa+\nu$
   (b) $\mu < \nu$ & $\kappa > 0 \Rightarrow \kappa.\mu < \kappa.\nu$
   (c) $\mu < \nu$ & $\kappa > 0 \Rightarrow \mu^{\kappa} < \nu^{\kappa}$
   (d) $\mu < \nu$ & $\kappa > 1 \Rightarrow \kappa^{\mu} < \kappa^{\nu}$

10. List the following cardinals in increasing order
    $2.\aleph_0$, $\aleph_0.\aleph_0$, $2^{\aleph_0}$, $\aleph_0^{\aleph_0}$, $2^{\aleph_1}$, $\aleph_1^{\aleph_0}$, $\aleph_0^{\aleph_1}$.

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PROBLEMS FOR CHAPTER 5 SPRING 2004

1. Evaluate (a) $\sum_{n \in \omega} n$ (c) $\prod_{n \in \omega} (n+1)$

   (b) $\sum_{\alpha \in \omega_1} |\alpha|$ (d) $\prod_{\alpha \in \omega_1} |\alpha + 1|$

2. Find two sequences of inf. cardinals $\langle \kappa_n \rangle$ and $\langle \mu_n \rangle$ such that $\kappa_n < \mu_n$ for each $n \in \mathbb{N}$ but yet $\sum_{n \in \mathbb{N}} \kappa_n = \sum_{n \in \mathbb{N}} \mu_n$

3. Let $\mathcal{P}_F(A)$ = set of all finite subsets of A. Prove that if A can be linearly ordered and $X \subseteq \mathcal{P}_F(A)$, then X has a choice function.

4. Prove that if A can be well-ordered, then $\mathcal{P}(A)$ can be linearly ordered.

5. Prove that if we can find a bijection from any set to an ordinal, then AC is true.

6. (a) Prove that $AC \Rightarrow DC_\omega$

   (b) Prove that $DC_\omega \Rightarrow AC_\omega$

7. (Optional) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$. Prove that if f is continuous, then $f(x) = ax$ for some $a \in \mathbb{R}$.

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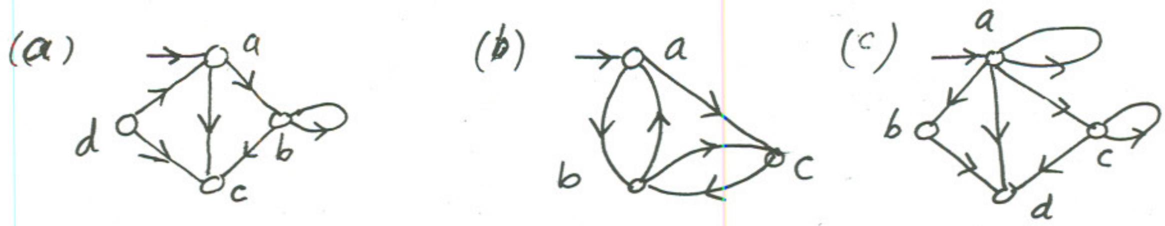
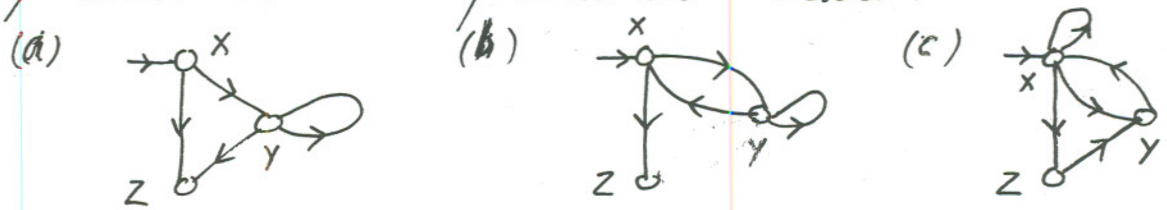
PROBLEMS FOR Ch. 6 — SPRING 2004

1. Find the reduced form of each of the following pseudo-sets



2. Find all the pseudo-sets that can be represented by a rooted digraph with ≤ 2 vertices.

3. Write down the system of equations satisfied by each of the pseudo-set below:



4. Find a pseudo-set which has infinitely many members but which is not well-founded.

5. Find a finite pseudo-set which needs a rooted digraph with infinitely many vertices to represent it.

6. Can a finite rooted digraph represent an infinite pseudo-set?