A1

MHF 4102 - AX. SET THEORY — FLORIDA INT'L UNIV.

PROBLEMS FOR CHAPTER 1 — SPRING 2004

1. Prove the following directly from the definitions
   (a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ (c) $X - Y = X - (X \cap Y)$
   (b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Y)$

2. If X and Y are both subsets of Z prove that
   (a) $Z - (X \cup Y) = (Z - X) \cap (Z - Y)$ (c) $Z - (Z - X) = X$
   (b) $Z - (X \cap Y) = (Z - X) \cup (Z - Y)$

3. Find (a) $\{\emptyset\} - \emptyset$ (d) $\{\{\emptyset\}\} - \{\emptyset\}$
   (b) $\{\emptyset, \{\emptyset\}\} - \{\{\emptyset\}\}$ (e) $\{\{\emptyset\}\} \cup \{\emptyset, \{\emptyset\}\}$
   (c) $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\}$ (f) $\{\emptyset\} - \{\{\emptyset\}\}$

4. If X and Y are sets, what are
   (a) $\bigcup \{X\}$ (c) $\bigcup \{X, Y\}$ (e) $\bigcup \emptyset$
   (b) $\bigcap \{X\}$ (d) $\bigcap \{X, Y\}$ (f) $\bigcap \emptyset$?

5. Let $\langle X_i : i \in I \rangle$ be a family of subsets of Z. Prove that
   (a) $Z - \bigcup_{i \in I} X_i = \bigcap_{i \in I} (Z - X_i)$ (b) $Z - \bigcap_{i \in I} X_i = \bigcup_{i \in I} (Z - X_i)$

6. Let R and S be the binary relations on $\mathbb{N}$ defined by
   aRb if a is a multiple of b
   aSb if a and b has no common factor.

   Determine whether or not R and S are
   (a) reflexive (b) symmetric (c) transitive
   (d) connected (e) anti-symmetric.

7. Let $A = \{1,2\}$. Enumerate all the binary relations on A. (Hint: There are 16 of them)

8. Let $A = \{a, b\}$ and $B = \{1,2,3\}$.
 (a) How many functions are there from A to B ?
 (b) How many of these functions are injections ?

9. (a) How many functions are there from $\{1,2\}$ to $\emptyset$
 (b) " " " " " $\emptyset$ to $\{1,2\}$
 (c) " " " " " $\emptyset$ to $\emptyset$

10. Let $f: X \to Y$ be a function, and A and B be subsets of Y. Prove that
 (a) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$
 (b) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$
 (c) $f^{-1}[A - B] = f^{-1}[A] - f^{-1}[B]$.

11. Let $f: X \to Y$ be a function and $A_i \subseteq Y$ for each $i \in I$. Which of the following are true
 (a) $f^{-1}\left[\bigcup_{i\in I} A_i\right] = \bigcup_{i\in I} f^{-1}[A_i]$
 (b) $f^{-1}\left[\bigcap_{i\in I} A_i\right] = \bigcap_{i\in I} f^{-1}[A_i]$?

12. Suppose $f: A \to B$ and $g: B \to C$ are both injective functions. Does it follow that $g \circ f: A \to C$ must also be an injective function.

MHF 4102 - AXIOMATIC SET THEORY SPRING 2004

PROBLEMS FOR CHAPTER 2 FLORIDA INT'L UNIV.

1. (a) Write down all the elements of V_4 in the cumulative hierarchy of sets.
   (b) Find a formula for the number of elements of V_n in terms of n only.

2. Write out each of the 10 axioms completely in the language of set theory. For example

   Nullset Axiom: $(\exists x_1)(\forall x_2)(\neg(x_2 \in x_1))$

   Extensionality Axiom:
   $$(\forall x_1)(\forall x_2)\Big(\big((\forall x_3)(x_3 \in x_1 \rightarrow x_3 \in x_2)\ \&\ (\forall x_4)(x_4 \in x_2 \rightarrow x_4 \in x_1)\big) \rightarrow (x_1 = x_2)\Big)$$

3. Let V = collection of all sets. Prove that V is a proper class.
   (Hint: Use the separation axiom and the fact that $R = \{x : x \notin x\}$ is a proper class.)

4. Let $\mathcal{A}$ be a class.
   (a) If $\mathcal{A}$ is a set prove that $\mathcal{A}^c$ is a proper class.
   (b) If $\mathcal{A}$ is a proper class does it follow that $\mathcal{A}^c$ is a set?

5. Prove that for any set A, $\mathcal{P}(A) \notin A$.
   (Hint: Let $D = \{a \in A : a \notin a\}$. Show that $D \in \mathcal{P}(A)$, but $D \notin A$.)

6. Use the Foundation axiom to show that there is no set x such that $x \in x$.
(Hint: Suppose x was a set such that $x \in x$. Let $A = \{x\}$. Show that A has no element a such that $a \cap A = \emptyset$.)

7. Let $\langle A, \leq \rangle$ be an ordered set. An element a is said to be a maximal element of A if there is no x in A with $a < x$.
Find the smallest ordered sets with
(a) 5 maximal elements and 3 minimal elements.
(b) 2 maximal elements and 4 minimal elements.

8. Let $\langle A, \leq \rangle$ be an ordered set such that any non-empty subset of A has a smallest element. Prove that "$\leq$" must be a linear ordering on A.

9. Let A be a non-empty set and R be the relation on A defined by
$$a R b \quad \text{if} \quad a \in b.$$
(a) Is it possible for R to be transitive?
(b) Is R always irreflexive?
(c) Is it possible for R to be symmetric?
(d) Is it possible for R to be connected?
(e) Is R always asymmetric?
Def. R is irreflexive if $a \not R a$ for all $a \in A$.

MHF 4102 AXIOMATIC SET THY. FLORIDA INT'L UNIV.

PROBLEMS FOR CHAPTER 3 SPRING 2004

1. Find a linearly ordered set $\langle L, < \rangle$ and an initial segment S of L such that S is not of the form $\{x : x < a\}$ with $a \in L$.

2. Find a linearly ordered set $\langle L, < \rangle$ and an increasing function $f : L \to L$ such that $f(x) > x$ for at least one $x \in L$ and $f(x) < x$ for at least one $x \in L$.

3. Explain why $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is not an ordinal

4. Prove that A is transitive if and only if $a \in A \Rightarrow a \subseteq A$.

5. Determine which of the following statements are true and which are not...
   (a) If X and Y are transitive, so is $X \cap Y$.
   (b) If X and Y are transitive, so is $X \cup Y$
   (c) If $X \in Y$ and Y is transitive, then X is transitive
   (d) If $X \subseteq Y$ and Y is transitive, then X is transitive
   (e) If every element in X is transitive, then X is transitive.

6. Prove that
   (a) If $\mathcal{A}$ is a set of ordinals, $\bigcup \mathcal{A}$ is an ordinal
   (b) If $\mathcal{A}$ is a <u>non-empty</u> set of ordinals, then $\bigcap \mathcal{A}$ is an ordinal.

(A6)

7. Recall that V_ω was defined inductively as follows:

$$V_0 = \emptyset$$
$$V_{n+1} = \mathcal{P}(V_n) \quad \text{for each } n \in \omega$$
$$V_\omega = \bigcup_{n\in\omega} V_n$$

Prove that

(a) V_ω is transitive

(b) $x \in V_\omega \Rightarrow \{x\} \in V_\omega$

(c) $x \in V_\omega \Rightarrow x \cup \{x\} \in V_\omega$.

8. (a) Prove the associative law for ordinal mult.

$$(\alpha . \beta) . \gamma = \alpha . (\beta . \gamma)$$

(b) Prove the following distributive law for ordinals

$$\alpha . (\beta + \gamma) = (\alpha . \beta) + (\alpha . \gamma)$$

9. Show that the following statements are not always true.

(a) If $\alpha + \gamma = \beta + \gamma$, then $\alpha = \beta$

(b) If $\gamma > 0$ and $\alpha . \gamma = \beta . \gamma$, then $\alpha = \beta$

(c) $(\beta + \gamma) . \alpha = (\beta . \alpha) + (\gamma . \alpha)$

10. Simplify:

(a) $(\omega + 2) + \omega$

(b) $\omega + \omega^2$

(c) $(\omega + 2) . (\omega + 3)$

(d) $(\omega + 1)^2 . \omega^3$

11. Prove that an ordinal α is a limit ordinal $\Leftrightarrow$ $\alpha = \omega . \beta$ for some ordinal β.

12. Find the smallest ordinal $\alpha \neq 0$ such that

(a) $\omega + \alpha = \alpha$ (b) $\omega . \alpha = \alpha$ (c) $\omega^\alpha = \alpha$.

MHF 4102 AXIOMATIC SET THEORY — FLORIDA INT'L UNIV.

PROBLEMS FOR CHAPTER 4 — SPRING 2004

1. Let $F(A,B)$ = set of all functions from A to B. Prove that for any sets A, B, & C
   (a) $A \preccurlyeq F(A,A)$
   (b) $A \subseteq B \Rightarrow F(A,C) \preccurlyeq F(B,C)$
   (c) $|B| \geq 2 \Rightarrow A \prec F(A,B)$
   (d) $B \subseteq C \Rightarrow F(A,B) \preccurlyeq F(A,C)$

2. Prove that
   (a) $\mathcal{P}(\mathbb{N}) \approx F(\mathbb{N},2)$ (Remember $2 = \{0,1\}$.)
   (b) $F(\mathbb{N},\mathbb{N}) \approx F(\mathbb{N},2)$

3. Prove that
   (a) $\mathbb{A} \approx \mathbb{N}$ ($\mathbb{A}$ = set of alg. nos.)
   (b) $\mathbb{R} \approx [0,1)$
   (c) $\mathbb{R} \approx F(\mathbb{N},2)$ <u>Hint</u>: Show that $[0,1) \approx F(\mathbb{N},2)$

4. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be an infinite sequence of 0's and 1's. We say that $\langle a_n \rangle$ has <u>finite support</u> if the number of 1's in $\langle a_n \rangle$ is finite. Prove that
   (a) the set of all sequences of 0's and 1's with finite support is equipotent to $\mathbb{N}$.
   (b) the set of all <u>finite</u> subsets of $\mathbb{N}$ is equipotent to $\mathbb{N}$.

5. Let $B(\mathbb{N},\mathbb{N})$ = set of all bijections from $\mathbb{N}$ to $\mathbb{N}$. Is $B(\mathbb{N},\mathbb{N}) \approx F(\mathbb{N},\mathbb{N})$?

MHF 4102 - AX. SET THY. FLORIDA INT'L UNIV.

PROBLEMS FOR CHAPTER 4 SPRING 2004

6. For any cardinal numbers κ, μ, ν prove that
   - (a) $\kappa + \mu = \mu + \kappa$
   - (b) $(\kappa + \mu) + \nu = \kappa + (\mu + \nu)$
   - (c) $\kappa + \kappa = 2 . \kappa$

7. For any cardinals numbers κ, μ, ν prove that
   - (a) $\kappa . \mu = \mu . \kappa$
   - (b) $(\kappa . \mu) . \nu = \kappa . (\mu . \nu)$
   - (c) $\kappa . (\mu + \nu) = \kappa . \mu + \kappa . \nu$

8. For any cardinal number κ, μ, ν prove that
   - (a) $\kappa . \kappa = \kappa^2$
   - (b) $\kappa^{\mu + \nu} = \kappa^{\mu} . \kappa^{\nu}$
   - (e) $(\kappa . \mu)^{\nu} = \kappa^{\nu} . \mu^{\nu}$
   - (d) $(\kappa^{\mu})^{\nu} = \kappa^{\mu . \nu}$

9. Determine which of the following are true
   - (a) $\mu < \nu \Rightarrow \kappa + \mu < \kappa + \nu$
   - (b) $\mu < \nu$ & $\kappa > 0 \Rightarrow \kappa . \mu < \kappa . \nu$
   - (c) $\mu < \nu$ & $\kappa > 0 \Rightarrow \mu^{\kappa} < \nu^{\kappa}$
   - (d) $\mu < \nu$ & $\kappa > 1 \Rightarrow \kappa^{\mu} < \kappa^{\nu}$

10. List the following cardinals in increasing order
    $2 . \aleph_0 , \quad \aleph_0 . \aleph_0 , \quad 2^{\aleph_0} , \quad \aleph_0^{\aleph_0} , \quad 2^{\aleph_1} , \quad \aleph_1^{\aleph_0} , \quad \aleph_n^{\aleph_1} .$

MHF 4102 - AX. SET THY. FLORIDA INT'L UNIV.

PROBLEMS FOR CHAPTER 5 SPRING 2004

1. Evaluate (a) $\sum_{n\in\omega} n$ (c) $\prod_{n\in\omega} (n+1)$

   (b) $\sum_{\alpha\in\omega_1} |\alpha|$ (d) $\prod_{\alpha\in\omega_1} |\alpha+1|$

2. Find two sequences of inf. cardinals $\langle \kappa_n \rangle$ and $\langle \mu_n \rangle$ such that $\kappa_n < \mu_n$ for each $n \in \mathbb{N}$ but yet $\sum_{n\in\mathbb{N}} \kappa_n = \sum_{n\in\mathbb{N}} \mu_n$

3. Let $\mathcal{P}_F(A)$ = set of all finite subsets of A. Prove that if A can be linearly ordered and $X \subseteq \mathcal{P}_F(A)$, then X has a choice function.

4. Prove that if A can be well-ordered, then $\mathcal{P}(A)$ can be linearly ordered.

5. Prove that if we can find a bijection from any set to an ordinal, then AC is true.

6. (a) Prove that $AC \Rightarrow DC_\omega$

   (b) Prove that $DC_\omega \Rightarrow AC_\omega$

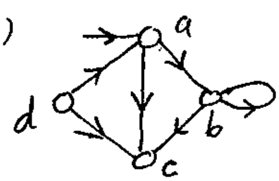
7. (Optional) Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$. Prove that if f is continuous, then $f(x) = ax$ for some $a \in \mathbb{R}$.

MHF 4102 - Ax. Set Th. FLORIDA INT'L UNIV.

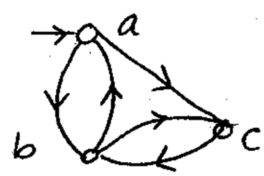
PROBLEMS FOR Ch. 6 SPRING 2004

1. Find the reduced form of each of the following pseudo-sets

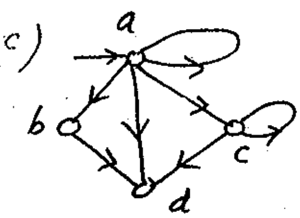
(a)



(b)



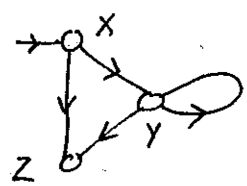
(c)



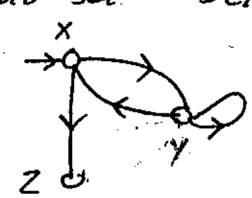
2. Find all the pseudo-sets that can be represented by a rooted digraph with ≤ 2 vertices.

3. Write down the system of equations satisfied by each of the pseudo-set below:

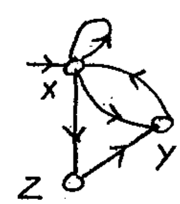
(a)



(b)



(c)



4. Find a pseudo-set which has infinitely many members but which is not well-founded.

5. Find a finite pseudo-set which needs a rooted digraph with infinitely many vertices to represent it.

6. Can a finite rooted digraph represent an infinite pseudo-set?

MHF 4102 - AXIOMATIC SET THEORY — FLORIDA INT'L UNIV.

SOLUTIONS OR HINTS TO PROBLEMS FROM Ch. 1. — SP'2004

1. (a) Suppose $a \in X \cup (Y \cap Z)$. Then $a \in X$ or $a \in Y \cap Z$.
Since $a \in Y \cap Z$ implies $a \in Y$, we have
$$a \in X \text{ or } a \in Y.$$
And since $a \in Y \cap Z$ implies $a \in Z$, we also have
$$a \in X \text{ or } a \in Z$$
Hence we have
$$a \in X \cup Y \quad \text{and} \quad a \in X \cup Z$$
Thus $a \in (X \cup Y) \cap (X \cup Z)$.
So $$X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z) \quad \ldots (1)$$

Suppose $a \in (X \cup Y) \cap (X \cup Z)$. Then
$a \in X \cup Y$ and
$a \in X \cup Z$.
Now if $a \notin X$, then we must have
$a \in Y$, because $a \in X \cup Y$
and $a \in Z$, because $a \in X \cup Z$.
So if $a \notin X$ then $a \in Y \cap Z$.
Hence we must have $a \in X$ or $a \in Y \cap Z$.
Thus $a \in X \cup (Y \cap Z)$.
So $$(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z) \quad \ldots (2)$$

From (1) & (2) it follows that
$$X \cup (Y \cap Z) = X \cup (Y \cap Z)$$

(b) The proof is very similar.

(B2)

1 (c) Suppose $a \in X-Y$. Then $a \in X$ and $a \notin Y$.
So $a \notin X \cap Y$ because $a \notin Y$.
Thus $a \in X$ and $a \in X \cap Y$.
$\therefore$ $a \in X-(X \cap Y)$.
Hence $X-Y \subseteq X-(X \cap Y)$ . . . (1)

Suppose $a \in X-(X \cap Y)$. Then $a \in X$ and $a \notin X \cap Y$. Now since $a \in X$, we must have $a \notin Y$ (otherwise we would get $a \in X \cap Y$)
So $a \in X$ and $a \notin Y$.
Thus $a \in X-Y$.
Hence $X - X \cap Y = X-Y$ . . . (2)

From (1) & (2) it follows that $X-Y = X - X \cap Y$.

2 (a) Suppose $a \in Z-(X \cup Y)$. Then
$a \in Z$ and $a \notin X \cup Y$.
Now if $a \notin X \cup Y$, then $a \notin X$ and $a \notin Y$ (because if a was in X or Y, a would be in $X \cup Y$). Hence
$a \in Z$, and $a \notin X$ and $a \notin Y$.
So $a \in Z$ and $a \notin X$,
Also $a \in Z$ and $a \in Y$.
Thus $a \in (Z-X)$ and $a \in (Z-Y)$
Hence $a \in (Z-X) \cap (Z-Y)$
$\therefore$ $Z-(X \cup Y) \subseteq (Z-X) \cap (Z-Y)$.

(2) Suppose $a \in (Z-X) \cap (Z-Y)$. Then

2 (a) ... $a \in Z-X$ and $a \in Z-Y$.
So $a \in Z$ and $a \notin X$, and
$a \in Z$ and $a \notin Y$.
We then have $a \in Z$. Also we have
$a \notin X$ and $a \notin Y$
which means that $a \notin X \cup Y$ (because if $a \notin X \cup Y$ we must have $a \in X$ or $a \in Y$).
Hence $a \in Z$ and $a \notin X \cup Y$.
Thus $a \in Z-(X \cup Y)$.
$\therefore \quad (Z-X) \cap (Z-Y) \subseteq Z-(X \cup Y)$

Hence $Z-(X \cup Y) = (Z-X) \cap (Z-Y)$.

(b) The proof is very, very similar.

(c) Let $a \in Z-(Z-X)$. Then $a \in Z$ and $a \notin (Z-X)$. Since $a \in Z$ we must also have $a \in X$ (otherwise if $a \notin X$ then we would get $a \in Z-X$). Hence $a \in X$. $\quad \therefore \quad Z-(Z-X) \subseteq X$.

Let $a \in X$. Then $a \in Z$ because $X \subseteq Z$. Now since $a \in Z$, we must have $a \notin Z-X$ (because if $a \in Z-X$, this would mean that $a \notin X$).
Hence $a \in Z$ and $a \notin (Z-X)$
$\therefore \quad a \in Z-(Z-X)$. So $X \subseteq Z-(Z-X)$
Hence $Z-(Z-X) = X$.

3. (a) $\{\emptyset\}$
   (b) $\{\emptyset\}$
   (c) $\{\emptyset\}$
   (d) $\{\{\emptyset\}\}$
   (e) $\{\emptyset, \{\emptyset\}\}$
   (f) $\{\emptyset\}$

4 (a) $\bigcup\{X\} = X$
  (b) $\bigcap\{X\} = X$
  (c) $\bigcup\{X, Y\} = X \cup Y$
  (d) $\bigcap\{X, Y\} = X \cap Y$
  (e) $\bigcup \emptyset = \emptyset$
  (f) $\bigcap \emptyset$ = collection of all sets.

5. (a) Let $a \in Z - \bigcup_{i\in I} A_i$. Then $a \in Z$ and $a \notin \bigcup_{i\in I} A_i$. So $a \notin A_i$ for each $i \in I$ because if a was in any of the A_i, a will be in the union $\bigcup_{i\in I} A_i$

Hence $a \in Z$ and $a \notin A_i$ for each $i \in I$.

Thus $a \in Z - A_i$ for each $i \in I$

$\therefore \quad a \in \bigcap_{i\in I}(Z - A_i)$.

So $\quad Z - \bigcup_{i\in I} A_i \subseteq \bigcap_{i\in I}(Z - A_i)$

Let $a \in \bigcap_{i\in I}(Z - A_i)$. Then for each $i \in I$ $a \in Z - A_i$.

So $a \in Z$ and $a \notin A_i$ for each $i \in I$.

Hence $a \in Z$ and $a \notin \bigcup_{i\in I} A_i$ because if a was in $\bigcup_{i\in I} A_i$, then would have to be in at least one A_i.

Thus $a \in Z - \bigcup_{i\in I} A_i$. $\therefore \bigcap_{i\in I}(Z - A_i) \subseteq Z - \bigcup_{i\in I} A_i$

Hence $Z - \bigcup_{i\in I} A_i = \bigcap_{i\in I}(Z - A_i)$

4

(b) The proof is very similar.

B5

6. (i) aRb if a is a multiple of b. $a,b \in \mathbb{N}$.

(a) R is reflexive because $a = 1.a$

(b) R is not symmetric because
$6R2$ but $2 \not R 6$

(c) R is transitive because if aRb & bRc
then $a = k.b$ and $b = l.c$
So $a = k.b = (kl).c$. Hence aRc.

(d) R is not connected because
$6 \not R 4$ and $4 \not R 6$

(e) R is anti-symmetric because
if $a \neq b$ and aRb then
$a > b$ or $a = 0$
and in either case case we see that
$b \not R a$.

(ii) aSb if a and b have no common factor
$a,b \in \mathbb{N}$.

(a) S is not reflexive because $2 \not S 2$.

(b) S is symmetric because if a and b have no common factor, then b and a have no common factor

(c) S is not transitive because
$6S5$ and $5S9$ but $6 \not S 9$.

(d) S is not connected because
$4 \not S 6$ and $6 \not S 4$.

(e) S is not anti-symmetric because
$2 \neq 3$ but $3S2$ and $2S3$

5

(B.6)

7. $A \times A = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$. A binary relation on A is just a subset of $A \times A$. Since there are 16 subsets of $A \times A$, there are 16 binary relations on A

8. (a) There are 9 functions from $\{a,b\}$ to $\{1,2,3\}$ (3 choices for $f(a)$, 3 choices for $f(b)$)
   (b) Six of these 9 functions are injections (3 choices for $f(a)$, only 2 choices for $f(b)$)

9. (a) 0 (There are no choices for $f(1)$ & $f(2)$)
   (b) 1 ($\emptyset$ is the only function from $\emptyset$ to $\{1,2\}$)
   (c) 1 ($\emptyset$ " " " " $\emptyset$ to $\emptyset$.)

10. (c) Let $a \in f^{-1}[A-B]$. Then by definition
$f(a) \in A - B$.
So $f(a) \in A$ and $f(a) \notin B$
$\therefore$ $a \in f^{-1}[A]$ and $a \notin f^{-1}[B]$
Thus $a \in f^{-1}[A] - f^{-1}[B]$
$\therefore$ $f^{-1}[A \cup B] \subseteq f^{-1}[A] - f^{-1}[B]$

Now let $a \in f^{-1}[A] - f^{-1}[B]$. Then
$a \in f^{-1}[A]$ and $a \notin f^{-1}[B]$
So $f(a) \in A$ and $f(a) \notin B$
$\therefore$ $f(a) \in A - B$. Thus $a \in f^{-1}[A-B]$
$\therefore$ $f^{-1}[A] - f^{-1}[B] \subseteq f^{-1}[A-B]$
Hence $f^{-1}[A-B] = f^{-1}[A] - f^{-1}[B]$

(6)

(a) (b) The proofs are very similar.

11. (a) TRUE. The proof is similar to the proof of (b)

(b) TRUE.

Let $a \in f^{-1}[\bigcap_{i\in I} A_i]$. Then

$f(a) \in \bigcap_{i\in I} A_i$.

So $f(a) \in A_i$ for each $i \in I$

$\therefore$ $a \in f^{-1}[A_i]$ for each $i \in I$

$\therefore$ $a \in \bigcap_{i\in I} f^{-1}[A_i]$.

$\therefore$ $f^{-1}[\bigcap_{i\in I} A_i] \subseteq \bigcap_{i\in I} f^{-1}[A_i]$.

Now let $a \in \bigcap_{i\in I} f^{-1}[A_i]$. Then

$a \in f^{-1}[A_i]$ for each $i \in I$

So $f(a) \in A_i$ for each $i \in I$

$\therefore$ $f(a) \in \bigcap_{i\in I} A_i$. So $a \in f^{-1}[\bigcap_{i\in I} A_i]$

$\therefore$ $\bigcap_{i\in I} f^{-1}[A_i] \subseteq f^{-1}[\bigcap_{i\in I} A_i]$

Hence $f^{-1}[\bigcap_{i\in I} A_i] = \bigcap_{i\in I} f^{-1}[A_i]$.

12. YES. Suppose $(g\circ f)(a_1) = (g\circ f)(a_2)$.
Then $g(f(a_1)) = g(f(a_2))$
So $f(a_1) = f(a_2)$ bec. g is injective.
Also we then get
$a_1 = a_2$ bec. f is injective.
Hence $(g\circ f)(a_1) = (g\circ f)(a_2)$ implies $a_1 = a_2$. Thus $g\circ f$ is injective.

AXIOMATIC SET THEORY — FLORIDA INT'L UNIV.

ANSWERS & HINTS TO PROBLEMS - Ch. 2 — SPRING 2004

1.(a) Let $0 = \emptyset$, $1 = \{0\}$, $\bar{1} = \{1\}$, $2 = \{0,1\}$ and $3 = \{0,1,2\}$. Then

$$V_4 = \{0,\ 1,\ \bar{1},\ \{\bar{1}\},\ \{2\},\ \{0,1,\bar{1},2\},\ 2,\ \{0,2\},\ \{0,\bar{1}\},\ \{1,\bar{1}\},\ \{1,2\},\ \{\bar{1},2\},\ 3,\ \{0,1,\bar{1}\},\ \{0,\bar{1},2\},\ \{1,\bar{1},2\}\}$$

(b) Let $\mathrm{tow}(2,n)$ be the function defined recursively as follows:

$$\mathrm{tow}(2,0) = 1$$
$$\mathrm{tow}(2,n+1) = 2^{\mathrm{tow}(2,n)}$$

Then V_n has $\mathrm{tow}(2,n-1)$ elements

$$\mathrm{tow}(2,n) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\} \; n \text{ two's here.}$$

2. <u>Ax. 1</u> $(\exists x_1)(\forall x_2)(\neg(x_2 \in x_1))$

<u>Ax. 2</u> $\Big(\big((\forall x_3)(x_3 \in x_1 \to x_3 \in x_2)\ \&\ (\forall x_4)(x_4 \in x_2 \to x_4 \in x_1)\big) \to (x_1 = x_2)\Big)$

<u>Ax. 3</u> $(\forall x_1)(\forall x_2)(\exists x_3)(\forall x_4)(x_4 \in x_3 \leftrightarrow (x_4 = x_1 \vee x_4 = x_2))$

<u>Ax. 4</u> $(\forall x_1)(\exists x_2)(\forall x_3)(x_3 \in x_2 \leftrightarrow (\exists x_4)(x_3 \in x_4\ \&\ x_4 \in x_1))$

(B9)

2. Ax.5 $(\forall x_1)(\exists x_2)(\forall x_3)\left(x_3 \in x_2 \leftrightarrow (\forall x_4)(x_4 \in x_3 \rightarrow x_4 \in x_1)\right)$

Ax.6 $(\exists x_1)\left(\phi \in x_1 \ \& \ (\forall x_2)(x_2 \in x_1 \rightarrow \{x_2\} \in x_1\right)$

Then replace "$\phi \in x_1$" by

$$(\exists x_3)\left(x_3 \in x_1 \ \& \ (\forall x_4)(\neg(x_4 \in x_3)\right)$$

and replace $\{x_2\} \in x_1$ by

$$(\exists x_5)\left(x_5 \in x_1 \ \& \ (\forall x_6)(x_6 \in x_5 \leftrightarrow x_6 = x_2)\right).$$

Ax.7 Who needs the aggravation!

$$(\forall x_1)\Big((\forall x_2)(\forall x_3)\big((x_2 \in x_1 \rightarrow x_2 \neq \phi) \ \& \ (x_2 \neq x_3) \ \& \ (x_2 \in x_1 \ \& \ x_3 \in x_1) \rightarrow (\forall x_4)(\neg(x_4 \in x_2 \ \& \ x_4 \in x_3)\big)$$
$$\rightarrow (\exists x_5)\Big((\forall x_6)(x_6 \in x_1 \rightarrow (\exists x_7)(x_7 \in x_5 \ \& \ x_7 \in x_6) \ \& \ (\forall x_8)((x_8 \neq x_7 \ \& \ x_8 \in x_6) \rightarrow x_8 \notin x_5\Big)\Big)\Big).$$ Oy!

Ax.8 $(\forall x_1)\left((\exists x_2)(x_2 \in x_1) \rightarrow (\exists x_3)(x_3 \in x_1 \ \& \ (\forall x_4)(\neg(x_4 \in x_1 \ \& \ x_4 \in x_3))\right)$

Ax.9 $(\forall x_1)(\exists x_2)(\forall x_3)\left(x_3 \in x_2 \leftrightarrow x_3 \in x_1 \ \& \ \varphi(x_3)\right)$

Since $\varphi(x_3)$ is a formula in L.O.S.T., the whole thing is in L.O.S.T.

Ax.10 $(\forall x_1)\Big(\big((\exists x_2)(\exists x_3)(\varphi(x_1,x_2) \ \& \ \varphi(x_1,x_3)) \rightarrow (x_2 = x_3)\big)$
$\rightarrow (\forall x_4)(\exists x_5)(\forall x_6)\left(x_6 \in x_5 \leftrightarrow (\exists x_7)\varphi(x_7,x_6)\right)\Big)$

Again since $\varphi(x_i, x_j)$ is a formula in L.O.S.T the whole thing is in L.O.S.T. (or should we say the whole thing is lost?)

By the way the answer for Ax.7 needs some minor fixing because $x_2 \neq \phi$, $x_8 \neq x_7$, and $x_8 \notin x_5$ are not allowed in LOST

<u>Ax. 5</u> $(\forall x_1)(\exists x_2)(\forall x_3)\left(x_3 \in x_2 \leftrightarrow (\forall x_4)(x_4 \in x_3 \rightarrow x_4 \in x_1)\right)$

<u>Ax. 6</u> $(\exists x_1)\left(\emptyset \in x_1 \;\&\; (\forall x_2)\left(x_2 \in x_1 \rightarrow \{x_2\} \in x_1\right)\right)$
This is not completely in L.O.S.T because "$\emptyset \in x_1$" and "$\{x_2\} \in x_1$" are not allowed in L.O.S.T.
But "$\emptyset \in x_1$" can be replaced by
$(\exists x_3)\left(x_3 \in x_1 \;\&\; (\forall x_4)(\neg(x_4 \in x_3))\right)$
And "$\{x_2\} \in x_1$" can be replaced by
$(\exists x_5)\left(x_5 \in x_1 \;\&\; (\forall x_6)\left(x_6 \in x_5 \leftrightarrow x_6 = x_2\right)\right)$
This makes every thing okay.

<u>Ax. 7</u> Who needs the aggravation!
$(\forall x_1)\Big(\big((\forall x_2)(\forall x_3) \rightarrow x_2 \neq \emptyset \;\&\; (\forall x_4)(\neg(x_4 \in x_2 \;\&\; x_4 \in x_3))\big)$
$\longrightarrow (\exists x_5)(\forall x_6)\big(x_6 \in x_1 \rightarrow (\exists x_7)(x_7 \in x_6 \;\&\; x_7 \in x_5 \;\&\; (\forall x_8)(x_8 \neq x_7$
$\rightarrow x_8 \notin x_5$

<u>Ax. 8</u> $(\forall x_1)\Big(x_1 \neq \emptyset \rightarrow (\exists x_2)\big(x_2 \in x_1 \;\&$
$(\forall x_3)\left(\neg\left(x_3 \in x_1 \;\&\; x_3 \in x_2\right)\right)\big)\Big)$
Of course "$x_1 \neq \emptyset$" is not allowed but we can replace it by
$(\exists x_4)(x_4 \in x_1)$

<u>Ax. 9</u> $(\forall x_1)(\exists x_2)(\forall x_3)\left(x_3 \in x_2 \leftrightarrow x_3 \in x_1 \;\&\; \varphi(x_3)\right)$
Since $\varphi(x_3)$ is a formula in L.O.S.T. the whole thing is in L.O.S.T.

<u>Ax. 10</u> $\varphi(x_5, x_6)$ is a function-type formula $\longrightarrow$
$(\forall x_1)(\exists x_2)(\forall x_3)\left(x_3 \in x_2 \leftrightarrow (\exists x_4)\; \varphi(x_4, x_3)\right)$
You must of course replace "$\varphi(x_5, x_6)$ is a function-type formula" by some thing completely in L.O.S.T.

3. First of all $V = \{x : x = x\}$. So V is a class. Now suppose that V is a set. Then by the separation axiom
$$\{x \in V : x \notin x\}$$
will also be a set. But
$$\{x \in V : x \notin x\} = \{x : x \notin x\} = R$$
which we know is a proper class (i.e. not a set). So we have a contradiction. Hence V is not a set.
So V is a proper class.

4. (a) Suppose $\mathcal{A}$ is a set. Now if $\mathcal{A}^c$ was also a set, then $\mathcal{A} \cup \mathcal{A}^c$ would be a set. But $\mathcal{A} \cup \mathcal{A}^c = V$ which is not a set. Hence $\mathcal{A}^c$ is not a set. To see that $\mathcal{A}^c$ is a proper class we just have to show that $\mathcal{A}^c$ is a class.
$$\mathcal{A}^c = \{x : x \notin \mathcal{A}\}$$
So $\mathcal{A}^c$ is clearly a class and we are done.

(b) NO. Let $\mathcal{A} = \{x : x \text{ has exactly one element}\}$ Then it can be shown that $\mathcal{A}$ and $\mathcal{A}^c$ are both proper classes.

B12

5. We want to show that $\mathcal{P}(A) \not\subseteq A$. Let

$$D = \{a \in A : a \notin a\}$$

Then D is clearly a subset of A, So $D \in \mathcal{P}(A)$.

Now suppose $D \in A$. Then D has a chance of being a member of D.
But if $D \in D$, then $D \notin D$ (contradiction) because D consists of all elements of A which are not members of themselves.
And if $D \notin D$, then $D \in D$ (contradiction again) because D consist of all elements of A which are not members of themselves.

So if we assume $D \in A$, we get a contradiction. Hence $D \notin A$. (If we assume $D \in A$, then D got a chance of being in D – that's what caused all the problems. If we assume $D \notin A$ we don't get any such problems because D does not get a chance of being in D. Only elements of A has a chance of being in D)

In any case, we now see that $D \in \mathcal{P}(A)$ but $D \notin A$. Hence $\mathcal{P}(A) \not\subseteq A$.

[Later on we will see that more than this is true. Cantor's theorem will tell us that $|\mathcal{P}(A)| > |A|$ and this clearly implies that $\mathcal{P}(A) \not\subseteq A$.]

B 13

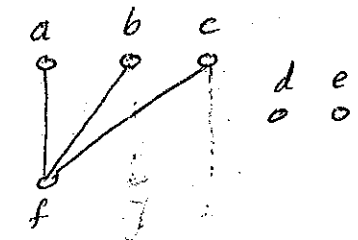
6. Suppose there is a set x such that $x \in x$. Let $A = \{x\}$. We want to show that A has no element a such that $a \cap A = \emptyset$. Since A has only one element, namely x, it is the only candidate which can give us an a such that $a \cap A = \emptyset$. But

$x \cap A \neq \emptyset$ because $x \in x$ and $x \in A$.

Hence there is element $a \in A$ such that $a \cap A = \emptyset$.

But this contradicts the Foundation axiom which says that if A is a non-empty set then we can find an element $a \in A$ such that $a \cap A = \emptyset$.

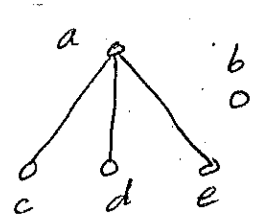
7. (a)



a, b, c, d, e are maximal elements

d, e, f are minimal elements

(b)



a and b are maximal elements

b, c, d, e are minimal elements

8. Since $\langle A, \leq \rangle$ is already an ordered set all we need to do to show that "d" is linearly ordered is to show that

$a \leq b$ or $b \leq a$ for any $a, b \in A$.

Suppose every non-empty subset of A has a smallest element. Let a and b be any two elements of A. Consider the set $\{a, b\}$. Since $\{a, b\} \neq \emptyset$ it has a smallest element. So

$a \leq b$, if a is the smallest

or $b \leq a$, if b is the smallest.

$\therefore$ $a \leq b$ or $b \leq a$.

Hence "$\leq$" is a linear ordering on A.

9. (a) YES, let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Then R is transitive on A.

(b) YES. For any set A, we must have $a \notin a$ for each $a \in A$ by Problem 6.

(c) YES. If A has only one element, then R will be symmetric on A by default

(d) YES. The set $A = \{\emptyset, \{\emptyset\}\}$ is connected under R

(e) YES. If A is a set and $a, b \in A$ and $a \in b$. Then $b \notin a$

MHF 4102 – AXIOMATIC SET THEORY — FLORIDA INT'L UNIV

SOLUTIONS OR HINTS TO PROBLEMS FOR Ch. 3 — SPRING 2004

1. Let $L = \mathbb{Q}$, the set of rational numbers, and "$<$" be the usual ordering in $\mathbb{Q}$. Take S to be the set of all non-positive numbers in $\mathbb{Q}$ i.e. $S = \{x \in \mathbb{Q} : x \leq 0\}$. Then S cannot be written in the form $\{x \in \mathbb{Q} : x < a\}$ for any $a \in \mathbb{Q}$ because there is no immediate successor of 0 in $\mathbb{Q}$.

2. Let $L = \mathbb{Z}$ and "$<$" be the usual ordering on $\mathbb{Z}$. Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n+2 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ n-2 & \text{if } n < 0 \end{cases}$$

Then f is increasing on $\mathbb{Z}$ but

$$f(1) = 3 > 1$$

and $f(-1) = -3 < -1$.

3. $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is not an ordinal because it is not strictly well-ordered by $\in$. The subset $\{\emptyset, \{\{\emptyset\}\}\}$ has no smallest element because $\emptyset$ and $\{\{\emptyset\}\}$ are not comparable.

<u>Note</u>: $\emptyset \in \{\emptyset\}$ and $\{\emptyset\} \in \{\{\emptyset\}\}$
but $\emptyset \notin \{\{\emptyset\}\}$.
Of course $\{\{\emptyset\}\} \notin \emptyset$ either.

4. (a) Suppose A is a transitive set. We want to show that $a \in A \Rightarrow a \subseteq A$.
So let a be an element of A.
Suppose x is an element of a. Then we have $x \in a$ and $a \in A$. Since A is transitive we get $x \in A$.
Thus if $x \in a$, then $x \in A$. $\therefore a \subseteq A$.
So $a \in A \Rightarrow a \subseteq A$.

(b) Now suppose $a \in A \Rightarrow a \subseteq A$. We want to show that A is transitive.
So let x and a be any sets such that $x \in a$ and $a \in A$. Since $a \in A \Rightarrow a \subseteq A$ we get $x \in a$ and $a \subseteq A$. So $x \in A$.
Thus $x \in a$ & $a \in A \Rightarrow x \in A$.
$\therefore$ A is a transitive set.

5. (a) TRUE (<u>Hint</u>: $x \in a$ & $a \in X \cap Y$ implies $x \in a$ & $a \in X$ and $x \in a$ & $a \in Y$. So $x \in X$ and $x \in Y$. $\therefore x \in X \cap Y$)

(b) TRUE (<u>Hint</u>: see hint above)

(c) FALSE. Let $X = \{\{\emptyset\}\}$ and $Y = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$
Then Y is transitive and $X \in Y$. but X is not transitive.

(d) FALSE. Let $X = \{\{\emptyset\}\}$ and $Y = \{\emptyset, \{\emptyset\}\}$
Then $X \subseteq Y$ & Y is transitive, but X is not

5. (e) FALSE Let $X = \{\{\emptyset\}\}$. Then every element of X is transitive but X is not.

6. (a) Let $\mathcal{A}$ be a set of ordinals. We want to show that $\cup\mathcal{A}$ is an ordinal. Let $X = \cup\mathcal{A}$.
Then $X = \{\beta : \beta \in \alpha \text{ for at least one } \alpha \text{ in } \mathcal{A}\}$

We first show that X is a transitive set.
Suppose $\gamma \in \beta$ and $\beta \in X$. Then we can find an ordinal α in $\mathcal{A}$ such that $\beta \in \alpha$.
So we have $\gamma \in \beta$ and $\beta \in \alpha$
Since α is an ordinal, we get $\gamma \in \alpha$.
So $\gamma \in X$ from the definition of X.
Hence X is a transitive set.

Now we know from Prop 5(c) of Ch. 3 (in class) that if α is an ordinal and $\beta \in \alpha$, then β is an ordinal. So X is a set of ordinal and so it follows from Thm 6 of Ch. 3 that $\langle X, \in \rangle$ is a strictly well-ordered set.
Thus X is an ordinal.

(b) Hint: Since $\mathcal{A}$ is non-empty, $\cap\mathcal{A}$ is a set. You can show that $X = \cap\mathcal{A}$ is a transitive set just as above.

$$X = \{\beta : \beta \in \alpha \text{ for every } \alpha \text{ in } \mathcal{A}\}$$

X will be a set of ordinals and so $\langle X, \in \rangle$ will be a strictly well-ordered set just as above.

7. (a) We have that $V_\omega = \bigcup_{n<\omega} V_n$.
Let $x \in V_\omega$.
If $x = \emptyset$ then $x \subseteq V_\omega$.
And if $x \neq \emptyset$, then $x \in V_{n+1}$ for some $n \in \mathbb{N}$.
But $V_{n+1} = \mathcal{P}(V_n)$. So $x \in \mathcal{P}(V_n)$ i.e. $x \subseteq V_n$.
Since $V_n \subseteq V_\omega$, it follows that $x \subseteq V_\omega$.

Thus $x \in V_\omega \Rightarrow x \subseteq V_\omega$. It now follows from Prob. #4 Ch. 3, that V_ω is transitive.

(b) <u>Hint</u>: $x \in V_\omega \Rightarrow x \in V_n$ for some $n \in \mathbb{N}$
$\Rightarrow \{x\} \subseteq V_n$
$\Rightarrow \{x\} \in V_{n+1}$
$\Rightarrow \{x\} \in V_\omega$

(c) <u>Hint</u>: See Hint above.

8. (b) We prove $\alpha.(\beta+\gamma) = (\alpha.\beta) + (\alpha.\gamma)$ by transfinite induction on γ. (α & β are parameters)

If $\gamma = 0$, then we have
$$\alpha.(\beta+\gamma) = \alpha.(\beta+0) = \alpha.\beta$$
$$= (\alpha.\beta) + 0 = (\alpha.\beta) + \gamma$$
So the result is true for 0.

Suppose the result is true for γ. We must prove it for $\gamma+1$. Now we have

8(b) $\alpha.(\beta+(\gamma+1)) = \alpha.((\beta+\gamma)+1)$ by def. of additio.

$= \alpha.(\beta+\gamma)+\alpha$ by def. of mult.

$= ((\alpha.\beta)+(\alpha.\gamma))+\alpha$ bec. result is true for γ

$= (\alpha.\beta)+((\alpha.\gamma)+\alpha)$ by Prop. 8

$= (\alpha.\beta)+(\alpha.(\gamma+1))$ by def. of mult.

as required.

Finally, suppose the result is true for all $\gamma<\lambda$ where λ is a limit ordinal. We must prove it for λ. We have

$\alpha.(\beta+\lambda) = \sup\{\alpha.(\beta+\gamma) : \gamma<\lambda\}$ by def. of mult.

$= \sup\{(\alpha.\beta)+(\alpha.\gamma) : \gamma<\lambda\}$ bec. result is true for all $\gamma<\lambda$

$= (\alpha.\beta)+\sup\{(\alpha.\gamma) : \gamma<\lambda\}$ bec. $\sup\{\alpha.\gamma : \gamma<\lambda\}$ is a limit ordinal

$= (\alpha.\beta)+(\alpha.\lambda)$ by def. of mult.

So the result is true for all γ. Since α and β were arbitrary the result is true for all α, β and γ.

8. (a) We will prove that $(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$ by transfinite induction on γ. [α and β will be arbitrary but fixed, i.e. parameters]

If $\gamma=0$, then we have

$(\alpha.\beta).\gamma = (\alpha.\beta).0 = 0$

$= \alpha.0 = \alpha.(\beta.0) = \alpha.(\beta.\gamma)$

So the result is true for $\gamma=0$.

8 (a) Now suppose the result is true for γ. We must prove it for $\gamma+1$. We have

$$\begin{aligned}(\alpha\cdot\beta)\cdot(\gamma+1) &= (\alpha\cdot\beta)\cdot\gamma + (\alpha\cdot\beta)\\ &= (\alpha\cdot(\beta\cdot\gamma)) + (\alpha\cdot\beta)\\ &= \alpha\cdot(\beta\cdot\gamma+\beta)\\ &= \alpha\cdot(\beta\cdot(\gamma+1))\end{aligned}$$

as required.

Finally suppose the result is true for all $\gamma<\lambda$ where λ is a limit ordinal. We must prove it for λ. We have

$$\begin{aligned}(\alpha\cdot\beta)\cdot\lambda &= \sup\{(\alpha\cdot\beta)\cdot\gamma : \gamma<\lambda\}\\ &= \sup\{\alpha\cdot(\beta\cdot\gamma) : \gamma<\lambda\}\\ &= \alpha\cdot\sup\{(\beta\cdot\gamma) : \gamma<\lambda\}\\ &= \alpha\cdot(\beta\cdot\lambda)\end{aligned}$$

So by the Transfinite Ind. Princ. the result is true for all γ. Since α & β were arb., it's also true for all α, β and γ.

9. (a), (b), & (c). These problems are part of PROJECT #1 and no more will be said about them. You are not allowed to discuss these problems with your classmates.

10. (a) $(\omega+2)+\omega = \omega+(2+\omega)$
$= \omega+\omega$
$= \omega.2$

(b) $\omega+\omega^2 = \omega.1+\omega.\omega$
$= \omega.(1+\omega)$
$= \omega.\omega = \omega^2$

(c) $(\omega+2)(\omega+3) = (\omega+2).\omega + (\omega+2).3$
$= (\omega+2)$ (ω times) $+ (\omega+2)$ (3 times)
$= \omega.\omega + (\omega.3+2)$
$= \omega^2+\omega.3+2$

(d) $(\omega+1)^2.\omega^3 = (\omega+1).(\omega+1).\omega^3$
$= [(\omega+1).\omega + (\omega+1).1].\omega^3$
$= [\omega.\omega + (\omega+1)].\omega^3$
$= [\omega^2+\omega+1].\omega^3$
$= \omega^2.\omega^3$
$= \omega^5$

11. (a) Suppose λ is a limit ordinal. Then $\lambda \leq \omega.\lambda$. Let β be the smallest ordinal α such that $\lambda \leq \omega.\alpha$. Then we can show that $\lambda = \omega.\beta$

Indeed, suppose β is a limit ordinal. Now if $\gamma<\beta$, then $\lambda > \omega.\gamma$ by the def. of β So $\lambda \geq \sup\{\omega.\gamma : \gamma<\beta\} = \omega.\beta$ So we will have $\lambda \geq \omega.\beta$

11 (a) And if $\beta = \gamma + 1$ then $\omega.\gamma < \lambda$ and $\omega.\beta$ would the next limit ordinal after $\omega.\gamma$. So since λ is a limit ordinal we must have $\lambda \geq \omega.\beta$

Thus in either case we get $\lambda \geq \omega.\beta$. Since $\omega\lambda \leq \omega.\beta$ by def. of β we get $\omega.\beta = \lambda$ as claimed

(b) The ordinal $\omega.\beta$ is clearly a limit ordinal because by the def. of mult

$$\omega.\beta = \omega \ (\beta \text{ times})$$
$$= \underbrace{\longrightarrow \longrightarrow \cdots \longrightarrow \cdots}_{\beta \text{ times}}$$

So there is no chance for $\omega.\beta$ to have a largest element. And this means that $\omega.\beta$ cannot be the successor of any ordinal.

12. (a) ω^2 $\qquad \omega + \omega^2 = \sup\{\omega + \omega.n : n < \omega\}$
$$= \sup\{\omega.(n+1) : n < \omega\} = \omega.\omega = \omega^2$$

(b) ω^ω $\qquad \omega.\omega^\omega = \sup\{\omega.\omega^n : n < \omega\}$
$$= \sup\{\omega^{n+1} : n < \omega\} = \omega^\omega$$

(c) ε $\qquad \omega^\varepsilon = \sup\{\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}} \dots\} = \varepsilon$

$$\varepsilon \underset{def}{=} \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}.$$

MHF 4102 - AX. SET THEORY FLORIDA INT'L UNIV.

SOLUTIONS OR HINTS TO Ch. 4 Problems SPRING '04

1. (a) For each $a \in A$, let f_a be the function defined by $f_a(x) = a$ for each $x \in A$.
Now define $j : A \to F(A,A)$ by
$j(a) = f_a$.
Then j is an injection, so $A \preccurlyeq F(A,A)$

(b) Let c be any element of C. For each function $f : A \to C$, let $f_c : B \to C$ be the function defined by

$$f_c(x) = \begin{cases} f(x) & \text{if } x \in A \\ c & \text{if } x \in B - A \end{cases}$$

Now define $j : F(A,C) \to F(B,C)$ by
$j(f) = f_c$.
Then j is an injection, so $F(A,C) \preccurlyeq F(B,C)$

(c) Let $B = \{b_0, b_1, \dots\}$. For each $a \in A$ define the function $f_a : A \to B$ by

$$f_a(x) = \begin{cases} b_0 & \text{if } x \neq a \\ b_1 & \text{if } x = a \end{cases}$$

Now define $j : A \to F(A,B)$ by
$j(a) = f_a$

Then j is an injection. So $A \preccurlyeq F(A,B)$

[We can actually show that $\mathcal{P}(A) \preccurlyeq F(A,B)$ if $|B| \geq 2$.

(d) This is very easy

B24

2. (a) For each subset A of $\mathbb{N}$, define a function $\chi_A : \mathbb{N} \to \{0,1\}$ by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{N} \\ 1 & \text{if } x \in \mathbb{N} \end{cases}$$

Now define $f: \mathcal{P}(\mathbb{N}) \to F(\mathbb{N},2)$ by $f(A) = \chi_A$. Then f is a bijection. So $\mathcal{P}(\mathbb{N}) \approx F(\mathbb{N},2)$.

(b) First observe that $F(\mathbb{N},2) \preccurlyeq F(\mathbb{N},\mathbb{N})$ by #1(d) because $2 = \{0,1\} \subseteq \mathbb{N}$.

We will prove that $F(\mathbb{N},\mathbb{N}) \preccurlyeq F(\mathbb{N},2)$. It will then follow that $F(\mathbb{N},\mathbb{N}) \approx F(\mathbb{N},2)$ by the Cantor-Bernstein theorem.

Let $\varphi: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be defined by $\varphi(\langle m,n\rangle) = 2^m(2n+1) - 1$. Then φ is a bijection. So the function

$$g : \mathcal{P}(\mathbb{N}\times\mathbb{N}) \to \mathcal{P}(\mathbb{N})$$

defined by

$$g(A) = \{\varphi(\langle m,n\rangle) : \langle m,n\rangle \in A\}$$

is a bijection.

Now a function $f: \mathbb{N} \to \mathbb{N}$ is just a set of ordered pairs; i.e., $f \subseteq \mathbb{N}\times\mathbb{N}$. So $F(\mathbb{N},\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$. Thus

$$F(\mathbb{N},\mathbb{N}) \preccurlyeq \mathcal{P}(\mathbb{N}\times\mathbb{N}) \approx \mathcal{P}(\mathbb{N}) \approx F(\mathbb{N},2)$$

$\therefore$ $F(\mathbb{N},\mathbb{N}) \preccurlyeq F(\mathbb{N},2)$ and we are done.

3 (a) This is problem #1 in Project #2. It can be proved by using the fact that a countable union of countable sets is countable. But this uses the Axiom of Choice. It can also be proved by using the Cantor Bernstein theorem (and in this approach the Axiom of Choice won't be used). No more can be said of this problem because it is a project.

(b) Let $f: \mathbb{R} \to [0,1)$ be defined by

$$f(x) = \begin{cases} 1/(2-x) & \text{if } x<0 \\ 1 - 1/(2+x) & \text{if } x \geqslant 0 \end{cases}$$

Then

f is an injection. (Actually f is a bijection from $\mathbb{R}$ to $(0,1)$. Just draw the graph & you'll see!)

So $\mathbb{R} \preccurlyeq [0,1)$.

Also since $[0,1) \subseteq \mathbb{R}$. So $[0,1 \preccurlyeq \mathbb{R}$.
Thus $\mathbb{R} \approx [0,1)$ by the Cantor-Bernstein theorem.

(c) We will show that $[0,1) \approx F(\mathbb{N},2)$.
First define $i: F(\mathbb{N},2) \to [0,1)$ by

$$i(f) = 0.f(0)f(1)f(2)f(3)\cdots \quad (\text{base } 10)$$

Then i is an injection. So

$$F(\mathbb{N},2) \preccurlyeq [0,1)$$

3 (c) Now recall that each real number has a unique decimal expansion in base 2 (an infinite tail of 1's is not allowed).

Define $j : [0,1) \to F(\mathbb{N}, 2)$ by
$j(x) = f$
where $f(n) = a_n$
if $x = 0.a_0 a_1 a_2 a_3 \ldots$ (in base 2)

Then j is an injection. So $[0,1) \preccurlyeq F(\mathbb{N},2)$
Thus $[0,1) \approx F(\mathbb{N}, 2)$.
Since $[0,1) \approx \mathbb{R}$, it follows that $\mathbb{R} \approx F(\mathbb{N},2)$

4. (a) Let SEQ_F = set of all infinite sequences of 0's & 1's with finite support.
Define $i : \mathbb{N} \to SEQ_F$ by
$i(n)$ = the inf. seq. with a "1" in just the n-th position only.
Then i is an injection. So $\mathbb{N} \preccurlyeq SEQ_F$

Also let $p_0, p_1, p_2, \ldots$ be the prime nos. in increasing order. Then define
$j : SEQ_F \to \mathbb{N}$ by
$j(\langle a_n \rangle) = p_0^{a_0} p_1^{a_1} p_2^{a_2} \ldots$
Then j is an injection. So $SEQ_F \preccurlyeq \mathbb{N}$.

Thus $SEQ_F \approx \mathbb{N}$.

4 (b) Let $\mathcal{P}_F(\mathbb{N})$ = set of all finite subsets of $\mathbb{N}$.
Define $f: \mathcal{P}_F(\mathbb{N}) \to SEQ_F$ by
$$f(A) = \langle \chi_A(0), \chi_A(1), \chi_A(2), \ldots \rangle$$
where
$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$
Then f is a bijection. So $\mathcal{P}_F(\mathbb{N}) \approx SEQ_F$.

5. Yes. First observe that $B(\mathbb{N},\mathbb{N}) \preccurlyeq F(\mathbb{N},\mathbb{N})$ because $B(\mathbb{N},\mathbb{N}) \subseteq F(\mathbb{N},\mathbb{N})$.

We say that a subset A of $\mathbb{N}$ is <u>co-finite</u> if $\mathbb{N}-A$ is finite. Let $\mathcal{P}_{CF}(\mathbb{N})$ be the set of all cofinite subsets of $\mathbb{N}$. Then clearly $\mathcal{P}_C(\mathbb{N}) \approx \mathcal{P}_F(\mathbb{N})$. Since $\mathcal{P}_F(\mathbb{N}) \approx \mathbb{N}$ we get that $\mathcal{P}_B(\mathbb{N}) \approx \mathcal{P}(\mathbb{N}) \approx F(\mathbb{N},\mathbb{N})$.

Here $\mathcal{P}_B(\mathbb{N}) = \mathcal{P}(\mathbb{N}) - \left(\mathcal{P}_F(\mathbb{N}) \cup \mathcal{P}_C(\mathbb{N})\right)$

Now define $j: \mathcal{P}_B(\mathbb{N}) \to B(\mathbb{N},\mathbb{N})$ by
$$j(A) = f_A$$
where
$$f_A(2k) = k\text{-th largest element of } A$$
$$f_A(2k+1) = k\text{-th largest element of } \mathbb{N}-A.$$
Then
j is an injection. So $\mathcal{P}_B(\mathbb{N}) \preccurlyeq B(\mathbb{N},\mathbb{N})$
Since $\mathcal{P}_B(\mathbb{N}) \approx F(\mathbb{N},\mathbb{N})$, $F(\mathbb{N},\mathbb{N}) \preccurlyeq B(\mathbb{N},\mathbb{N})$
Thus $B(\mathbb{N},\mathbb{N}) \approx F(\mathbb{N},\mathbb{N})$.

6. (a) $\kappa+\mu = |\kappa \times \{0\} \cup \mu \times \{1\}|$

$\approx |\mu \times \{0\} \cup \kappa \times \{1\}|$

$= \mu + \kappa$

$\therefore \kappa+\mu = \mu+\kappa$

(b) $(\kappa+\mu)+\nu = |(\kappa+\mu) \times \{0\} \cup \nu \times \{1\}|$

$\approx |(\kappa \times \{0\} \cup \mu \times \{1\}) \times \{0\} \cup \nu \times \{1\}|$

$= |\kappa \times \{\langle 0,0\rangle\} \cup \mu \times \{\langle 1,0\rangle\} \cup \nu \times \{1\}|$

$\approx |\kappa \times \{0\} \cup \mu \times \{\langle 0,1\rangle\} \cup \nu \times \{\langle 1,1\rangle\}|$

$\approx |\kappa \times \{0\} \cup (\mu+\nu) \times \{1\}|$

$= \kappa + (\mu+\nu)$.

(c) $\kappa + \kappa = |\kappa \times \{0\} \cup \kappa \times \{1\}|$

$= |\kappa \times \{0,1\}|$

$\approx |\{0,1\} \times \kappa| = 2 \cdot \kappa$

7. (a) $\kappa \cdot \mu = |\kappa \times \mu| \approx |\mu \times \kappa| = \mu \cdot \kappa$

(b) $(\kappa \cdot \mu) \cdot \nu = |(\kappa \times \mu) \times \nu| \approx |\kappa \times (\mu \times \nu)| = \kappa \cdot (\mu \cdot \nu)$

(c) $\kappa \cdot (\mu+\nu) = |\kappa \times (\mu+\nu)|$

$\approx |\kappa \times (\mu \times \{0\} \cup \nu \times \{1\})|$

$\approx |(\kappa \times \mu) \times \{0\} \cup (\kappa \times \nu) \times \{1\}|$

$\approx |(\kappa \cdot \mu) \times \{0\} \cup (\kappa \cdot \nu) \times \{1\}|$

$= \kappa \cdot \mu + \kappa \cdot \nu$.

(B29)

8. (a) For each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ let $f_{\alpha,\beta} : 2 \to \kappa$ be the function defined by

$$f_{\alpha,\beta}(0) = \alpha, \quad f_{\alpha,\beta}(1) = \beta$$

(Remember $2 = \{0,1\}$)

Now define $j : \kappa \times \kappa \to F(2, \kappa)$ by

$$j(\langle \alpha, \beta \rangle) = f_{\alpha,\beta}.$$

Then j is a bijection.

So

$$\kappa . \kappa = |\kappa \times \kappa| \approx |F(2, \kappa)| = \kappa^2$$

(b)

$$\begin{aligned} \kappa^{\mu+\nu} &= |F(\mu+\nu, \kappa)| \\ &\approx |F(\mu,\kappa) \times F(\nu, \kappa)| \\ &= |F(\mu,\kappa)| . |F(\nu,\kappa)| \\ &= \kappa^{\mu} . \kappa^{\nu} \end{aligned}$$

(c) For each function $f : \nu \to \kappa \times \mu$ we can get two functions $f_1 : \nu \to \kappa$ and $f_2 : \nu \to \mu$ as follows:

If $f(\alpha) = \langle \beta, \gamma \rangle$, let $f_1(\alpha) = \beta$ and $f_2(\alpha) = \gamma$.

The function $j : F(\nu, \kappa \times \mu) \to F(\nu, \kappa) \times F(\nu, \mu)$ defined by $j(f) = \langle f_1, f_2 \rangle$ is a bijection.

So

$$\begin{aligned} (\kappa . \mu)^{\nu} &= |F(\nu, \kappa . \mu)| \\ &= |F(\nu, \kappa \times \mu)| \\ &\approx |F(\nu, \kappa) \times F(\nu, \mu)| \\ &\approx |F(\nu, \kappa)| \cdot |F(\nu, \mu)| = \kappa^{\nu} . \mu^{\nu} \end{aligned}$$

(B30)

8 (d) Let $f: \mu \times \nu \to \kappa$ be a function. For each value $\beta_0 \in \nu$ we can get a function $f_{\beta_0}: \mu \to \kappa$ by letting

$$f_{\beta_0}(\alpha) = f(\langle \alpha, \beta_0 \rangle) \qquad \alpha \in \mu.$$

Let $\varphi_f: \nu \to F(\mu, \kappa)$ be defined by $\varphi_f(\beta) = f_\beta$ and define $j: F(\mu \times \nu, \kappa) \to F(\nu, F(\mu, \kappa))$ by

$$j(f) = \varphi_f.$$

Then j is a bijection.

So $\quad F(\mu \times \nu, \kappa) \approx F(\nu, F(\mu, \kappa))$

$$\begin{aligned} \text{Thus} \quad (\kappa^\mu)^\nu &= |F(\nu, \kappa^\mu)| \\ &= |F(\nu, F(\mu, \kappa))| \\ &\approx |F(\mu \times \nu, \kappa)| \\ &\approx |F(\mu . \nu, \kappa)| \\ &= \kappa^{\mu . \nu} \end{aligned}$$

Hence $(\kappa^\mu)^\nu = \kappa^{\mu . \nu}$.

9. (a) $2 < 3$ but $\aleph_0 + 2 \not< \aleph_0 + 3$

(b) $2 < 3$ & $\aleph_0 > 0$ but $\aleph_0 . 2 \not< \aleph_0 . 3$

(c) $2 < 3$ & $\aleph_0 > 0$ but $2^{\aleph_0} \not< 3^{\aleph_0}$

(d) $2 < 3$ & $\aleph_0 > 1$ but $\aleph_0^2 \not< \aleph_0^3$

B 31

10. $2 \cdot \aleph_0 = \max(2, \aleph_0) = \aleph_0$

$\aleph_0 \cdot \aleph_0 = \max(\aleph_0, \aleph_0) = \aleph_0$

$\aleph_0 < 2^{\aleph_0}$ by Cantor's Diagonal Theorem.

$\aleph_0^{\aleph_0} = 2^{\aleph_0}$ by Qu. 2(b)

$\aleph_1 < 2^{\aleph_1}$ by Cantor's diagonal Theorem

$2^{\aleph_0} \leq 2^{\aleph_1}$ because $\aleph_0 \leq \aleph_1$.

$$2^{\aleph_0} \leq (\aleph_1)^{\aleph_0} \leq \left(2^{\aleph_0}\right)^{\aleph_0} \quad \text{bec. } \aleph_1 \leq 2^{\aleph_0}$$
$$= 2^{\aleph_0 \cdot \aleph_0}$$
$$= 2^{\aleph_0}$$

So $\aleph_1^{\aleph_0} = 2^{\aleph_0}$

$$2^{\aleph_1} \leq (\aleph_0)^{\aleph_1} \leq (\aleph_2)^{\aleph_1} \leq \left(2^{\aleph_1}\right)^{\aleph_1} \quad \text{bec. } \aleph_2 \leq 2^{\aleph_1}$$
$$= 2^{\aleph_1 \cdot \aleph_1}$$
$$= 2^{\aleph_1}$$

So $(\aleph_2)^{\aleph_1} = (\aleph_0)^{\aleph_1} = 2^{\aleph_1}$

So $2 \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0 < 2^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph_1^{\aleph_0}$
$\leq 2^{\aleph_1} = \aleph_0^{\aleph_1} = (\aleph_2)^{\aleph_1}$

MHF 4102 AX. SET TH. — FLORIDA INT'L UNIV.

SOLUTIONS OR HINTS FOR Ch. 5 PROBLEMS — SPRING 2004

1. (a) $$\aleph_0 \le \sum_{n\in\omega} n \le \sum_{n\in\omega} \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0$$
$$\therefore \sum_{n\in\omega} n = \aleph_0$$

(b) $$\aleph_1 \le \sum_{\alpha\in\omega_1} |\alpha| \le \sum_{\alpha\in\omega_1} \aleph_0 = \aleph_0 \cdot \aleph_1 = \aleph_1$$
$$\therefore \sum_{\alpha\in\omega_1} |\alpha| = \aleph_1$$

(c) $$2^{\aleph_0} \le \prod_{n\in\omega} (n+1) \le \prod_{n\in\omega} \aleph_0 = \aleph_0^{\aleph_0} = 2^{\aleph_0}$$
$$\therefore \prod_{n\in\omega} (n+1) = 2^{\aleph_0}$$

(d) $$2^{\aleph_1} \le \prod_{n\in\omega_1} |\alpha+1| \le \prod_{n\in\omega_1} \aleph_0 = \aleph_0^{\aleph_1} = 2^{\aleph_1}$$
$$\therefore \prod_{n\in\omega_1} |\alpha+1| = 2^{\aleph_1}.$$

2. Let $\kappa_n = \aleph_n$, for $n \in \mathbb{N}$ and
$\mu_n = \aleph_{2^n}$, for $n \in \mathbb{N}$.
Then $\kappa_n < \mu_n$ for each $n \in \mathbb{N}$.
But
$$\aleph_\omega \le \sum_{n\in\mathbb{N}} \kappa_n \le \sum_{n\in\mathbb{N}} \mu_n \le \sum_{n\in\mathbb{N}} \aleph_\omega = \aleph_\omega \cdot \aleph_0 = \aleph_\omega$$
So
$$\sum_{n\in\mathbb{N}} \kappa_n = \sum_{n\in\mathbb{N}} \mu_n = \aleph_\omega.$$

3. Let X be any subset of $\mathcal{P}_F(A)$ and suppose A can be linearly ordered. We can define a choice function
$$f: X \to \cup X$$
as follows. First fix a lin. ordering "$\prec$" on A

If $\emptyset \in X$, let $f(\emptyset) = \emptyset$, and if $\mathcal{B} \neq \emptyset$ let $f(\mathcal{B})$ = smallest element in $\mathcal{B}$ according to "$\prec$". Since $\mathcal{B}$ is finite we can always find a smallest element. (If $\mathcal{B}$ were infinite we might not be able to do so because $\prec$ was just a linear ordering, not a well-ordering.)

Thus we are able to find a choice function for X

4. Suppose A can be well-ordered. Let us fix a well-ordering "$\prec$" on A. We want to show that $\mathcal{P}(A)$ can be linearly ordered.

Let X and Y be elements of $\mathcal{P}(A)$ with $X \neq Y$

Then $(X-Y) \cup (Y-X) \neq \emptyset$.

So $(X-Y) \cup (Y-X)$ has a smallest element, a say, according to "$\prec$".

Put $X <_L Y$ if $a \in Y-X$

and $Y <_L X$ if $a \in X-Y$

33 Then $\langle \mathcal{P}(A), <_L \rangle$ will be a linearly ordered set.

<u>Note</u>: "$<_L$" is an extension of the partial ordering "$\subset$" (proper subset

5. Suppose that we can find a bijection from any set to an ordinal. Let $\mathcal{A}$ be any set of pairwise disjoint non-empty sets. To show that AC is true, we must find a set M which consists of exactly one element from each member of $\mathcal{A}$.

Since $\mathcal{A}$ is a set, $B = \cup \mathcal{A}$ is also a set. So we can find a bijection $f : B \to \beta$. Now for each $A \in \mathcal{A}$, let

$$\Gamma = \{ f(a) : a \in A \}$$

Since Γ is a non-empty set of ordinals it has a smallest element, α_A say.

Let $M = \{ f^{-1}(\alpha_A) : A \in \mathcal{A} \}$. Then M consists of exactly one element of each member of A (bec. $f^{-1}(\alpha_A) \in A$). So we are done.

[Basically the ordinal β induces a well-ordering "$<$" on the set B. From each set $A \in \mathcal{A}$ we pick the smallest element according to this well-ordering "$<$". This will give us the required M.]

6.(a) $AC \Rightarrow DC_\omega$:

Suppose AC is true. Let A be a set and u be an element of A. Also let R be a relation on A such that for any $x \in A$, there is a $y \in A$ such that xRy

We must show that there is a sequence $\langle z_n : n \in \omega \rangle$ of elements of A such that

$$z_0 = u, \text{ and}$$
$$z_n R z_{n+1} \quad \text{for all } n \in \omega.$$

Since AC is true, we can find a choice function f on $\mathcal{P}(A)$. Define a function $s: \omega \to A$ by recursion as follows:

$$s(0) = u, \text{ and}$$
$$s(n+1) = f(\{a \in A : s(n) R a\})$$

Since $\{a \in A : s(n) R a\} \neq \emptyset$, $s(n+1) \in A$ and $s(n) R s(n+1)$.

So if we put $z_n = s(n)$ we are done

(b) AC_ω is really the statement : If $\mathcal{A}$ is a denumerable set of pairwise disjoint nonempty sets, then there is a set M which consists of exactly one element of each member of $\mathcal{A}$.

Suppose DC_ω is true. Let $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ be a denumerable set of pairwise disjoint

6(b) non-empty sets. Then by logic we can find a member u of A_0. Now define the relation R on $B = \cup \mathcal{A}$ by

$$x R y \quad \text{if} \quad x \in A_n \text{ and } y \in A_{n+1}.$$

for some $n \in \omega$.

So if $x \in A_0$ & $y \in A_1$, then $x R y$
But if $x \in A_0$ & $y \in A_2$, then $x \not R y$
And if $x \in A_1$ & $y \in A_0$, then $x \not R y$.

Since DC_ω is true we can find a sequence $\langle z_n : n \in \omega \rangle$ such that

$$z_0 = u \quad \text{and}$$
$$z_n R z_{n+1} \quad \text{for each } n \in \omega.$$

Let $M = \{z_n : n \in \omega\}$. Then M will contain exactly one ~~element~~ element of each member of $\mathcal{A}$.

MHF 4102 - Ax. Set 7h. FLORIDA INT'L UNIV.

SOLUTIONS OR HINTS TO Ch.6 PROBLEMS SPRING 2004

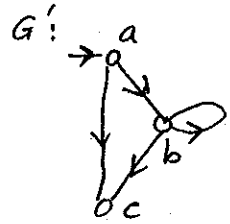
1. (a) d is inaccessible from the root a

P_0 : $\{a, b, c\}$

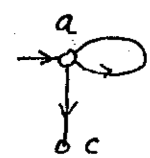
P_1 : $\{a, b\}\ \{c\}$

P_2 : $\{a, b\}\ \{c\} = P_1$

G':



G^R:

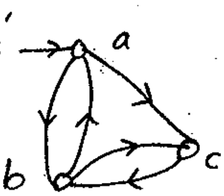


(b) No inaccessible vertices from a

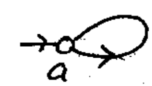
P_0 : $\{a, b, c\}$

P_1 : $\{a, b, c\} = P_0$

G':



G^R:



(c) No inaccessible vertices from a.

P_0 : $\{a, b, c, d\}$

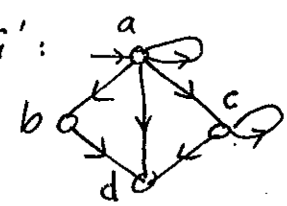
P_1 : $\{a, b, c\}\ \{d\}$

P_2 : $\{a, c\}\ \{b\}\ \{d\}$

P_3 : $\{a\}\{c\}\{b\}\ \{d\}$

P_4 : $= P_3$.

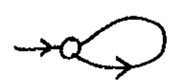
G':



$G^R := G'$

2. (a) $0 = \to\circ$

(b) θ

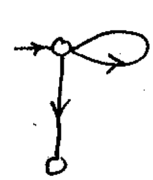


$\theta = \{\theta\}$

(c)

$= 1$

(d) p



$p = \{0, p\}$

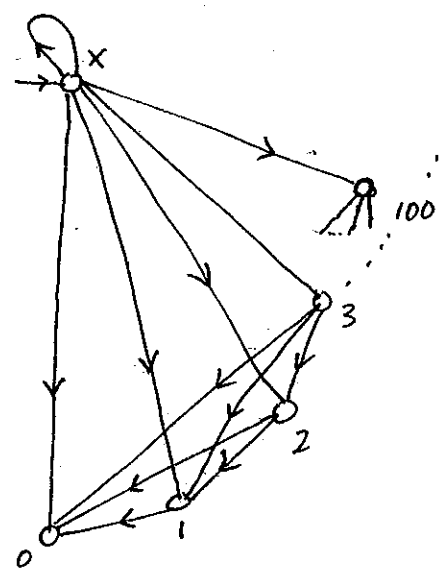
These are the only ones. Every rooted digraph with ≤ 2 vertices can be reduced to one of these four.

3. (a) $x = \{y, z\}$, $y = \{y, z\}$, $z = \emptyset$

(b) $x = \{y, z\}$, $y = \{x, y\}$, $z = \emptyset$

(c) $x = \{x, y, z\}$, $y = \{x\}$, $z = \{y\}$

4.

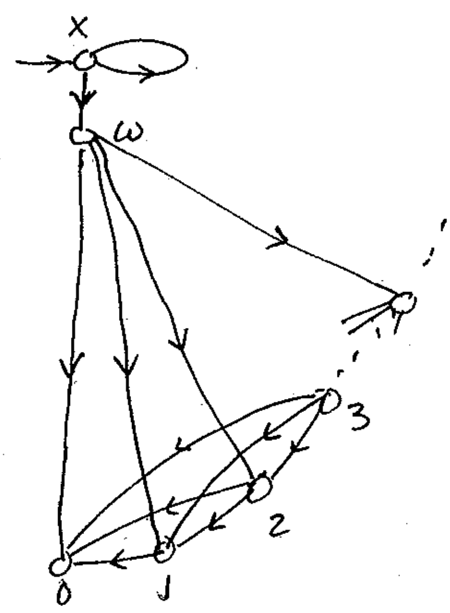


$x = \{x, 0, 1, 2, 3, \ldots, 100, \ldots\}$

x is infinite because $\omega \subseteq x$.

x is not well-founded because $x \in x$.

5.



$x = \{\omega, x\}$

x is finite because x has only two elements: ω & x.
($\omega \neq x$ bec. $x \in x$ but $\omega \notin \omega$)

x cannot be represented by a finite rooted digraph because $\omega \in x$ and ω needs an infinite rooted digraph.

6. No. If $\langle G, a\rangle$ is a finite rooted digraph, then the no. of edges out of the root gives an upper bound for the no. of elements in the pseudo-set rep. by $\langle G, a\rangle$. Since G is finite, outdeg (a) is finite. So the pseudo-set is finite