C14

MHF 4102 - Ax. Set Theory — Florida Int'l Univ.

TEST # 1 - Spring 2004 — TIME: 75 min.

Answer all 6 questions. Justify all of your answers.

(15) 1. Let $\langle B_i \rangle_{i \in I}$ be a family of subsets of $\mathbb{R}$. Prove that $\mathbb{R} - \left(\bigcap_{i \in I} B_i\right) = \bigcup_{i \in I} (\mathbb{R} - B_i)$

(20) 2 (a) Define ordinal multiplication by using transfinite recursion

(b) Prove that for any ordinals α, β, γ $\alpha.(\beta+\gamma) = \alpha.\beta + \alpha.\gamma$.

(You may use the fact that ordinal addition is associative.)

(20) 3 (a) Write down the Separation Axiom & the Replacement Axiom in both their ordinary & class forms.

(b) Write down the Power Set Axiom & the Foundation Axiom in their ordinary form. Then traslate them into the language of set theory, (LOST).

(15) 4. Let V = collection of all sets & $S = \{x : x$ has one element$\}$

(a) Using the fact that $R = \{x : x \notin x\}$ is not a set, prove that V & S are not sets.

(b) Prove that S is a class.

(15) 5. (a) Define what is a well-ordered set $\langle A, < \rangle$.

(b) If $\langle A, < \rangle$ is a well-ordered set and $f : A \to A$ is an increasing function, prove that $x \leq f(x)$ for all $x \in A$.

(15) 6 (a) Define what is a transitive set & what is an ordinal.

(b) Suppose R is reflexive and aRb & $bRc \Rightarrow cRa$. Prove that R is an equivalence relation on A.

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Solutions to Test #1 — Spring 2004

1. Let $x \in \mathbb{R} - (\bigcap_{i \in I} B_i)$. Then $x \in \mathbb{R}$ and $x \notin \bigcap_{i \in I} B_i$
So $x \in \mathbb{R}$ and $\neg (\forall i \in I)(x \in B_i)$.
So $x \in \mathbb{R}$ and $(\exists i \in I)(x \notin B_i)$
$\therefore$ $(\exists i \in I)(x \in \mathbb{R}$ and $x \notin B_i)$
$\therefore$ $(\exists i \in I)(x \in \mathbb{R} - B_i)$. $\therefore$ $x \in \bigcup_{i \in I}(\mathbb{R} - B_i)$
So $\mathbb{R} - \bigcap_{i \in I} B_i \subseteq \bigcup_{i \in I}(\mathbb{R} - B_i)$

Now let $x \in \bigcup_{i \in I}(\mathbb{R} - B_i)$. Then $(\exists i \in I)(x \in \mathbb{R} - B_i)$
So $x \in \mathbb{R}$ and $x \notin B_i$ for some $i \in I$
$\therefore$ $x \in \mathbb{R}$ and $(\exists i \in I)(x \notin B_i)$
$\therefore$ $x \in \mathbb{R}$ and $\neg (\forall i \in I)(x \in B_i)$
$\therefore$ $x \in \mathbb{R}$ and $x \notin \bigcap_{i \in I} B_i$. So $x \in \mathbb{R} - \bigcap_{i \in I} B_i$
Hence $\bigcup_{i \in I}(\mathbb{R} - B_i) \subseteq \mathbb{R} - \bigcap_{i \in I} B_i$

Thus $\mathbb{R} - (\bigcap_{i \in I} B_i) = \bigcup_{i \in I}(\mathbb{R} - B_i)$

2. (a) For each ordinal β we define $\beta . \gamma$ by transfinite recursion on γ as follows:
(i) $\beta . 0 = 0$, (ii) $\beta . (\gamma + 1) = (\beta . \gamma) + \beta$, and
(iii) $\beta . \lambda = \sup\{\beta . \gamma : \gamma < \lambda\}$ if λ is a limit ordinal.

(b) We will prove that $\alpha . (\beta + \gamma) = \alpha . \beta + \alpha . \gamma$ by Transfinite induction on γ. Suppose $\gamma = 0$. Then

$$\alpha . (\beta + 0) = \alpha . \beta = \alpha . \beta + 0$$
$$= \alpha . \beta + \alpha . 0$$

So the result is true for $\gamma = 0$

2. Suppose the result is true for γ. We will prove it for $\gamma+1$. Now

$$\begin{aligned}
\alpha.(\beta+(\gamma+1)) &= \alpha.((\beta+\gamma)+1) && \text{bec. addition is assoc.}\\
&= \alpha.(\beta+\gamma)+\alpha && \text{by (ii) above}\\
&= (\alpha.\beta+\alpha.\gamma)+\alpha && \text{result true for } \gamma\\
&= \alpha.\beta+(\alpha.\gamma+\alpha) && \text{addition is assoc.}\\
&= \alpha.\beta+\alpha.(\gamma+1) && \text{by (ii) above}
\end{aligned}$$

Finally suppose the result is true for all $\gamma < \lambda$ where λ is a limit ordinal. We will prove it for λ.

$$\begin{aligned}
\alpha.(\beta+\lambda) &= \sup\{\alpha.(\beta+\gamma) : \gamma<\lambda\} && \text{bec. } \beta+\lambda \text{ is limit ord.}\\
&= \sup\{\alpha.\beta+\alpha.\gamma : \gamma<\lambda\} && \text{result true for } \gamma<\lambda\\
&= \alpha.\beta+\sup\{\alpha.\gamma : \gamma<\lambda\}\\
&= \alpha.\beta+\alpha.\lambda
\end{aligned}$$

So by the Transfinite Induction Principle, the result is true for all γ. Since α & β were arbitrary, it is true for all α, β and γ.

3(a) <u>Sep. Axiom</u>: If $\varphi(x)$ is any formula of L.O.S.T. and A is a set then $\{x \in A : \varphi(x) \text{ is true}\}$ is a set

<u>Class Form</u>: If $\mathcal{B}$ is a class & A is a set then $\mathcal{B} \cap A$ is a set.

<u>Repl. Axiom</u>: If $\varphi(x,y)$ is any function-type formula of L.O.S.T. and A is a set, then $\{b : \varphi(a,b) \text{ is true for at least one } a \in A\}$ is a set.

<u>Class Form</u>: If $\mathcal{F}$ is any class-function and A is a set, then $\mathcal{F}[A] = \{\mathcal{F}(a) : a \in A\}$ is a set.

(b) <u>Power Set Axiom</u>: If A is a set then $\mathcal{P}(A)$ is a set

3 (b) $(\forall x_1)(\exists x_2)(\forall x_3)\big(x_3 \in x_2 \leftrightarrow (\forall x_4)(x_4 \in x_3 \to x_4 \in x_1)\big)$.

<u>Foundation Axiom:</u> If A is any non-empty set, there exists $x \in A$ such that $x \cap A = \emptyset$.

$$(\forall x_1)\big((\exists x_2)(x_2 \in x_1) \to (\exists x_3)(\,x_3 \in x_1 \;\&\; (\forall x_4)\ \neg((x_4 \in x_1) \,\&\, (x_4 \in x_3)))\big)$$

4.(a) Suppose V is a set. Then by the separation Axiom, $\{x \in V : x \notin x\}$ will be a set. But $\{x \in V : x \notin x\} = \{x : x \notin x\} = R$ is not a set – so we have a contradiction. Hence V is not a set. Now suppose S is a set. The by the union axiom, $\cup S$ will be a set. But $\cup S = \cup\{\{x\} : x \text{ is a set}\} = \{x : x \text{ is a set}\} = V$ which is not a set – so we have another contradiction. Hence S is not a set.

(b) We just have to find a formula of L.O.S.T. which says the same thing as "x has one element". Let $\varphi(x)$ be the formula
$(\exists x_1 \in x) \wedge (\forall x_2)(\forall x_3)(x_2 \in x \,\&\, x_3 \in x \to x_2 = x_3)$.
Then $S = \{x : \varphi(x) \text{ is true}\}$ and so is a class.

5. (a) A well-ordered set $\langle A, < \rangle$ is a partially ordered set $\langle A, < \rangle$ in which any non-empty subset has a smallest element.

(b) Recall that $f: A \to A$ is increasing if $x < y \Rightarrow f(x) < f(y)$. Now suppose $\neg(\forall x \in A)(x \leq f(x))$. Let $B = \{x \in A : f(x) < x\}$. Then $B \neq \emptyset$. So B has a smallest element, b say.

5 (b) Let $b' = f(b)$. Then $b' < b$ because $f(b) < b$. So $f(b') < f(b)$ because f is increasing. $\therefore f(b') < b'$. $\therefore b' \in B$ & $b' < b$. But b was the smallest element of B, so we have a contradiction. Hence $(\forall x \in A)(x \leq f(x))$.

6 (a) A set S is transitive if $a \in b$ & $b \in S \Rightarrow a \in S$. An ordinal is a transitive set A such that $\langle A, \in \rangle$ is a strictly well-ordered set.

(b) Recall that an equivalence relation is one which is reflexive, transitive and symmetric. Now we already know that R is reflexive.

Suppose aRb. Then bRb because R is reflexive. Hence aRb and bRb. Since aRb & $bRc \Rightarrow cRa$, it follows that bRa. So R is symmetric.

Finally suppose aRb and bRc. Then cRa from what was given. But we just showed R was symmetric. Hence aRc. So aRb and $bRc \Rightarrow aRc$. Thus R is transitive.

Hence R is an equivalence relation.