MHF 4102 - AX. SET THEORY FLORIDA INT'L UNIV.

TEST #1 - Spring '07 TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each question on a separate page.

(15) 1. Let $\langle A_i : i \in I \rangle$ be an indexed family of subsets of the set X. Prove that

$$\bigcup_{i \in I} (X - A_i) = X - \left(\bigcap_{i \in I} A_i \right).$$

(15) 2 Let $f : X \to Y$ be a function and suppose that $A, B \subseteq X$ and $C, D \subseteq Y$.

(a) Is it always true that $f[A] \cap f[B] \subseteq f[A \cap B]$?

(b) Is it always true that $f^{-1}[C] \cap f^{-1}[D] \subseteq f^{-1}[C \cap D]$?

(20) 3(a) Write down the *Separation and Replacement axioms* in both their ordinary form and in class-form.

(b) Write down the *Union and Foundation axioms* in their ordinary form. Then translate them completely into the language of set theory.

(15) 4(a) Using the fact that $R = \{x : x \notin x\}$ is not a set, prove that $V = \{x : x = x\}$ is not a set.

(b) Prove that the collection S_2 of all sets with exactly two element is a proper class.

(15) 5(a) Define what is a *minimal element* and what is a *minimum element* of a partially ordered set $(A, <)$.

(b) Define what it means for $(A, <)$ to be *well-founded*. Using the Foundation Axiom prove that there is no set x such that $x \in x$.

(20) 6(a) Define *ordinal addition* and *ordinal multiplication*.

(b) Prove that if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$ for all ordinals α, β, γ by transfinite induction.

(c) Hence show that if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.

MHF 4102 - Axiomatic Set Theory — Florida Int'l Univ.
Solutions to Test #1 — Spring 2007

1. Let $a \in \bigcup_{i\in I}(X-A_i)$. Then $a \in (X-A_{i_0})$ for at least one $i_0 \in I$. So $a \in X$ and $a \notin A_{i_0}$. Since $a \notin A_{i_0}$, $a \notin \bigcap_{i\in I} A_i$. Hence $a \in X - \bigcap_{i\in I} A_i$. $\therefore \bigcup_{i\in I}(X-A_i) \subseteq X - \bigcap_{i\in I} A_i$.

Now let $a \in X - \bigcap_{i\in I} A_i$. Then $a \in X$ and $a \notin \bigcap_{i\in I} A_i$. So $a \in X$ and $a \notin A_{i_0}$ for at least one $i_0 \in I$. Hence $a \in (X - A_{i_0})$. So $a \in \bigcup_{i\in I}(X-A_i)$. $\therefore X - \bigcap_{i\in I} A_i \subseteq \bigcup_{i\in I}(X-A_i)$.
Thus $\bigcup_{i\in I}(X-A_i) = X - (\bigcap_{i\in I} A_i)$.

2 (a) NO. Let $f: \mathbb{Z} \to \mathbb{Z}$; $f(z) = |z|$. Take $A = \{0,1,2,3,\ldots\}$ and $B = \{0,-1,-2,-3,\ldots\}$. Then $A \cap B = \{0\}$ and $f[A] \cap f[B] = \{0,1,2,3,\ldots\} \cap \{0,1,2,3,\ldots\} = \{0,1,2,3,\ldots\}$ while $f[A\cap B] = \{f(0)\} = \{0\} \not\supseteq f[A] \cap f[B]$.

(b) YES. Let $x \in f^{-1}[C] \cap f^{-1}[D]$. Then $x \in f^{-1}[C]$ and $x \in f^{-1}[D]$. So $f(x) \in C$ and $f(x) \in D$. $\therefore f(x) \in C \cap D$ $\therefore x \in f^{-1}[C \cap D]$. So $f^{-1}[C] \cap f^{-1}[D] \subseteq f^{-1}[C\cap D]$ always.

3 (a) Separation Axiom: If $\varphi(x)$ is any formula of L.O.S.T. with one free variable x and A is any set, then $\{x \in A : \varphi(x)\}$ is a set. (Class-form): If $\mathcal{C}$ is a class and A is a set, then $\mathcal{C} \cap A$ is a set.
Replacement Axiom: If $\varphi(x,y)$ is any function-type formula of L.O.S.T. and A is a set, then $\{b : \varphi(a,b)$ holds for at least one b in $A\}$ is a set. (Class-form): If $\mathcal{F}$ is a class-function and A is a set, the $\mathcal{F}[A]$ is a set.

3(b) Union Axiom: If A is a set, then $\cup A$ is a set, i.e., For any set A there exists a set B such that for all b, $b \in B$ if and only if there is a with $b \in a$ and $a \in A$.

$$(\forall x_1)(\exists x_2)(\forall x_3)\ (x_3 \in x_2 \leftrightarrow (\exists x_4)(x_3 \in x_4 \wedge x_4 \in A))$$

Foundation Axiom: If A is a non-empty set, then A has a minimal element w.r.t. "$\in$", i.e., for any set A, if $A \neq \emptyset$, then there exists a set a such that $a \in A$ and $a \cap A = \emptyset$.

$$(\forall x_1)\ ((\exists x_2)(x_2 \in x_1) \rightarrow (\exists x_3)(x_3 \in x_1 \wedge (\forall x_4)\ \neg(x_4 \in x_3 \wedge x_4 \in x_1))).$$

4(a). Suppose V was a set. Then $\{x \in V : x \notin x\}$ would be a set by the Separation Axiom. But $\{x \in V : x \notin x\}$ $= \{x : x = x \wedge x \notin x\} = \{x : x \notin x\} = R$ which is not a set. So we have a contradiction. $\therefore$ V is not a set.

(b) First, let us show that S_2 is a class. We have

$$S_2 = \{\{x_1, x_2\} : x_1 \in V,\ x_2 \in V \text{ and } x_1 \neq x_2\}$$
$$= \{x_3 : \varphi(x_3) \text{ holds}\}$$ where $\varphi(x_3)$ is the formula

$$(\exists x_1)(\exists x_2)(((x_1 \in x_3 \wedge x_2 \in x_3) \wedge \neg(x_1 = x_2) \wedge$$
$$(\forall x_4)\ (x_4 \in x_3 \rightarrow (x_4 = x_1 \vee x_4 = x_2)))$$

Hence S_2 is a class because $\varphi(x_3)$ is a formula of LOST

Now suppose S_2 was a set. Then by the union Axiom, $\cup S_2$ would also be a set. But

$$\cup S_2 = \cup\{\{x_1, x_2\} : x_1 \neq x_2 \text{ and } x_1, x_2 \in V\}$$
$$= \{x_1 : x_1 \in V\} \cup \{x_2 : x_2 \in V\} = V \cup V = V$$

But we know by part (a) that V is not a set, so we have a contradiction. Hence S_2 is not a set. So S_2 is a proper class.

5 (a) A minimal element of $(A,<)$ is any element $b \in A$ such that there is no $c \in A$ with $c < b$. A minimum (or smallest) element of $(A,<)$ is any element $b \in A$ such that $c < b$ for all $c \in A - \{b\}$.

(b) The partially ordered set $(A,<)$ is well-founded if every non-empty subset B of A has a minimal element.

Suppose x is a set with $x \in x$. Let $A = \{x\}$. Then A is a non-empty set. So by the Foundation Axiom, $(A, \in)$ has a minimal element, i.e., $\exists a \in A$ such that $a \cap A = \emptyset$. Since A has only one element a must be x. But $x \in x$ and $x \in A$, so $x \in x \cap A$. Hence $x \cap A \neq \emptyset$ and so A has no minimal element, a contradiction. Hence there is no set x with $x \in x$.

6 (a) Ordinal addition and multiplication are defined by transfinite recursion on β as follows:

(i) $\alpha + 0 = \alpha$, $\alpha + (\beta+1) = (\alpha+\beta)+1$, and
$\alpha + \lambda = \sup\{\alpha+\beta : \beta < \lambda\}$.

(ii) $\alpha . 0 = 0$, $\alpha . (\beta+1) = (\alpha . \beta) + \alpha$ and
$\alpha . \lambda = \sup\{\alpha . \beta : \beta < \lambda\}$.

(b) We will prove that $\beta < \gamma \Rightarrow \alpha + \beta < \alpha + \gamma$ by transfinite induction on γ. If $\gamma = 0$, then there is no β with $\beta < \gamma$, so the result is vacuously true.

If $\gamma = 1$, then β must be 0. Now $\alpha + 0 = \alpha < \alpha + 1$ So $\alpha + \beta < \alpha + \gamma$. Hence the result is true for $\gamma = 1$.

6(b) Suppose the result is true for γ. Let $\beta < \gamma+1$.
Then $\beta < \gamma$ or $\beta = \gamma$. Now if $\beta < \gamma$, then

$$
\begin{aligned}
\alpha+\beta &< \alpha+\gamma && \text{bec. result is true for } \gamma \\
&< (\alpha+\gamma)+1 && \text{bec. } \delta < \delta+1 \text{ for any } \delta. \\
&= \alpha+(\gamma+1) && \text{by the definition of ord. "+"}
\end{aligned}
$$

And if $\beta = \gamma$, then $\alpha+\beta = \alpha+\gamma < \alpha+(\gamma+1)$.
So if the result is true for γ, it follows for $\gamma+1$.

Finally suppose the result is for all $\gamma < \lambda$ with λ a lim.ord.
Let $\beta < \lambda$. Then $\beta < \gamma_0$ for some $\gamma_0 < \lambda$ because λ is a limit ordinal. So

$$
\begin{aligned}
\alpha+\beta &< \alpha+\gamma_0 && \text{bec. result is true for } \gamma_0 \\
&\leq \sup\{\alpha+\gamma : \gamma < \lambda\} = \alpha+\lambda
\end{aligned}
$$

So if the result is true for all γ with $\gamma < \lambda$, then it follows for λ.

By the Transfinite Induction Principle it follows that the result is true for all γ.

(c) Assume that $\alpha+\beta = \alpha+\gamma$.
Now if $\beta < \gamma$, then $\alpha+\beta < \alpha+\gamma$ by part (b)
And if $\gamma < \beta$, then $\alpha+\gamma < \alpha+\beta$ by part (b) also.
So the only way for us to have $\alpha+\beta = \alpha+\gamma$ is to have $\beta = \gamma$.