MHF 4102 - AX. SET THEORY | FLORIDA INT'L UNIV.

TEST #2 - Spring 2007 | TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit.

(20) 1. (a) Define what is a non-trivial *limit ordinal* and define *ordinal exponentiation*.

(b) Prove that for any ordinals α, β, γ we have $\alpha^{\beta+\gamma} = \alpha^{\beta}.\alpha^{\gamma}$. [You may use any results you need about ordinal multiplication and ordinal addition.]

(15) 2. Simplify (a) $4 + (\omega.3) + \omega^2$ (Ordinal arithmetic)

(b) $(\omega.2 + 3)^2$ (Ordinal arithmetic)

as far as possible.

(20) 3. (a) Define what are *continuous* and *increasing* class-functions from Ω to Ω. [Here Ω = class of all ordinals]

(b) Suppose that $f:\Omega \to \Omega$ is a *continuous* and *increasing* class-function. Prove that we can always find an *infinite ordinal* β such that $f(\beta) = \beta$.

(15) 4. (a) Give the definitions of the following *cardinal operations*: $\kappa+\mu$, $\kappa.\mu$, κ^{μ}

(b) If κ is a *cardinal*, prove that $\kappa.\kappa = \kappa^2$.

(15) 5. (a) Define what is a *choice function* for a set A.

(b) Suppose the set A can be well-ordered. Prove in ZF that the power set of A can then be *linearly ordered*.

(15) 6. (a) Define what is a *cardinal* κ and what is the *cofinality* of a limit ordinal λ.

(b) Simplify $\aleph_1.2^{\aleph_1} + \aleph_2.2^{\aleph_0}$ (Cardinal arithmetic) as far as possible.

MHF 4102 - AXIOMATIC SET THEORY FLORIDA INT'L UNIV.

SOLUTIONS TO TEST # 2 SPRING 2007

1.(a) A non-trivial limit ordinal is any ordinal $\alpha > 0$ such that there exists no ordinal β with $\alpha = \beta + 1$.
Ordinal exponentiation is defined transfinite recursion as follows $\alpha^0 = 1$, $\alpha^{\beta+1} = \alpha^\beta . \alpha$, and $\alpha^\lambda = \sup\{\alpha^\gamma : 0 < \gamma < \lambda\}$ if λ is a non-triv. limit ord.

(b) We will prove that $\alpha^{\beta+\gamma} = \alpha^\beta . \alpha^\gamma$ by transfinite induction on γ. If $\gamma = 0$, then

$$\begin{aligned}
\alpha^{\beta+0} &= \alpha^\beta && \text{because } \beta+0=0 \\
&= \alpha^\beta . 1 && \text{because } \delta = \delta . 1 \\
&= \alpha^\beta . \alpha^0 && \text{because } \alpha^0 = 1
\end{aligned}$$

Now suppose the result is true for γ. Then

$$\begin{aligned}
\alpha^{\beta+(\gamma+1)} &= \alpha^{(\beta+\gamma)+1} && \text{bec. } \beta+(\gamma+1) = (\beta+\gamma)+1 \\
&= \alpha^{\beta+\gamma} . \alpha && \text{by the def. of ord exp.} \\
&= (\alpha^\beta . \alpha^\gamma) . \alpha && \text{bec result is true for } \gamma \\
&= \alpha^\beta . (\alpha^\gamma . \alpha) && \text{be ord. mult. is assoc.} \\
&= \alpha^\beta . \alpha^{\gamma+1} && \text{by the def. of ord. exp.}
\end{aligned}$$

So if the result is true for γ, it will be true for $\gamma+1$.
Finally, suppose the result is true for all $\gamma < \lambda$. Then

$$\begin{aligned}
\alpha^{\beta+\lambda} &= \sup\{\alpha^{\beta+\gamma} : \gamma < \lambda\} && \text{bec. } \beta+\lambda \text{ is a limit ord.} \\
&= \sup\{\alpha^\beta . \alpha^\gamma : \alpha < \lambda\} && \text{bec. result is true for } \gamma. \\
&= \alpha^\beta . \sup\{\alpha^\gamma : \alpha < \lambda\} && \text{bec. } \sup\{\alpha^\gamma : \alpha < \lambda\} \text{ is a lim. ord.} \\
&= \alpha^\beta . \alpha^\lambda && \text{by the def. of ord. exp.}
\end{aligned}$$

So if the result is true for all $\gamma < \lambda$, it will be true for λ. Hence by the Transfinite induction principle, the result is true for all γ. Since α & β were fixed but arbitrary, it is all true for all α and β.

2(a) $4 + \omega.3 + \omega^2 = 4 + (\omega + \omega + \omega) + \omega^2$

$= (4 + \omega) + \omega + \omega + \omega^2$

$= \sup\{4+n : n<\omega\} + \omega + \omega + \omega^2$

$= \omega + \omega + \omega + \omega^2$

$= \omega.3 + \omega.\omega = \omega.(3+\omega)$

$= \omega.\sup\{3+k : k<\omega\} = \omega.\omega = \omega^2$

(b) $(\omega.2+3)^2 = (\omega.2+3).(\omega.2+3)$

$= (\omega.2+3).(\omega.2) + (\omega.2+3).3$ left distr.

$= [(\omega+\omega+3).\omega].2 + (\omega+\omega+3).3$

$= \omega^2.2 + \omega.6 + 3$ because

$(\omega+\omega+3).3 = (\omega+\omega+3) + (\omega+\omega+3) + (\omega+\omega+3)$

$= \omega + \omega + (3+\omega) + \omega + (3+\omega) + \omega + 3$

$= \omega + \omega + \omega + \omega + \omega + \omega + 3 = \omega.6 + 3$

& $(\omega+\omega+3).\omega = \sup\{(\omega+\omega+3).n : n<\omega\}$

$= \sup\{\underbrace{(\omega+\omega+3) + \cdots + (\omega+\omega+3)}_{n \text{ times}} : n<\omega\}$

$= \sup\{\underbrace{\omega+\omega+\cdots+\omega}_{2n \text{ times}} + 3 ; n<\omega\}$

$= \sup\{\omega.(2n)+3 : n<\omega\} = \omega^2$ because

$\omega^2 = \sup\{\omega.(2n) : n<\omega\} \leq \sup\{\omega.(2n)+3 \; n<\omega\}$

$\leq \sup\{\omega.(2n+1) : n<\omega\} = \omega^2$

3(a) A class-function $f : \Omega \to \Omega$ is continuous if for every non-trivial limit ordinal λ, $f(\lambda) = \sup\{f(\alpha) : \alpha < \lambda\}$.

A class function $f : \Omega \to \Omega$ is increasing if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$.

3 (b) Since f is increasing we know by a theorem in class that $f(\alpha) \geq \alpha$ for all $\alpha \in \Omega$. Now if $f(\omega) = \omega$, take $\beta = \omega$ and we are done. Otherwise, define the sequence $\langle \beta_n \rangle_{n \in \omega}$ as follows. Let $\beta_0 = \omega$ and for $n \geq 0$, put $\beta_{n+1} = f(\beta_n)$. Since f is increasing we have

$$\omega < f(\beta_0) < f(f(\beta_0)) < \cdots\cdots < f^{(n)}(\beta_0) < \cdots$$

$$\therefore \beta_0 < \beta_1 < \beta_2 < \cdots\cdots < \beta_n < \cdots$$

Let $\beta = \sup\{\beta_n : n < \omega\}$. We claim that β is a non-trivial limit ordinal. Suppose $\beta = \gamma + 1$. Then for some n_0, we must have $\beta_{n_0} \geq \gamma + 1$ (otherwise we would have $\beta = \sup\{\beta_n : n < \omega\} \leq \gamma$). But then $\beta_{n_0+1} = f(\beta_{n_0}) > \gamma + 1$ which would contradict the fact that $\beta = \gamma + 1$. Hence β is a non-trivial limit ordinal. Since f is continuous, we have $f(\beta) = \sup\{f(\alpha) : \alpha < \beta\} = \sup\{f(\beta_n) : n < \omega\} = \beta$. So in either case we found an infinite ordinal β with $f(\beta) = \beta$.

4 (a) $\kappa + \mu = |\kappa \times \{0\} \cup \kappa \times \{1\}|$,
$\kappa \cdot \mu = |\kappa \times \mu|$, and $\kappa^\mu = |\mathcal{F}(\mu, \kappa)|$.

(b) Let $g : \kappa \times \kappa \to \mathcal{F}(2, \kappa)$ be the function defined by $g(\langle \alpha, \beta \rangle) = f_{\alpha,\beta}$ where $f_{\alpha,\beta} : 2 \to \kappa$ is the function given by $f_{\alpha,\beta}(0) = \alpha$ and $f_{\alpha,\beta}(1) = \beta$. Then g is a bijection. So

$$\begin{aligned} \kappa \cdot \kappa &= |\kappa \times \kappa| \\ &= \mathcal{F}(2, \kappa) \qquad \text{bec. } \kappa \times \kappa \approx \mathcal{F}(2, \kappa) \\ &= \kappa^2 \end{aligned}$$

5(a) A choice function for A is any function f with domain A such that $f(x) \in x$ for each non-empty set x in A.

(b) Let $<$ be a well-ordering on A. We define a relation $\prec$ on $\mathcal{P}(A)$ as follows. Let X and Y be any two distinct elements of $\mathcal{P}(A)$. Then $X \neq Y$ and $X \subseteq A$ & $Y \subseteq A$. So $(X-Y) \cup (Y-X) \neq \emptyset$. Since $\langle A, < \rangle$ is a well-ordered set, $(X-Y) \cup (Y-X)$ has a smallest element, a_0 say. Put

$$X \prec Y \quad \text{if} \quad a_0 \in Y$$

and $Y \prec X$ if $a_0 \in X$.

Then $\langle \mathcal{P}(A), \prec \rangle$ will be a linearly ordered set. The smallest element of $\mathcal{P}(A)$ (under $\prec$) will be $\emptyset$ and the largest will be A. "$\prec$" basically extends the subset relation into a linear ordering.

6(a) A cardinal κ is any ordinal κ such that for all $\alpha < \kappa$, $\alpha \not\approx \kappa$.

The cofinality of λ is the smallest ordinal θ such that there is a sequence $\langle \alpha_\beta : \beta < \theta \rangle$ in λ with $\sup\{\alpha_\beta : \beta < \theta\} = \lambda$.

(b) $\aleph_1 \cdot 2^{\aleph_1} + \aleph_2 \cdot 2^{\aleph_0}$

$= \max(\aleph_1, 2^{\aleph_1}) + \max(\aleph_2, 2^{\aleph_0})$ (Thm. on Card mult.)

$= 2^{\aleph_1} + \max(\aleph_2, 2^{\aleph_0})$ bec. $\aleph_1 < 2^{\aleph_1}$

$= \max(2^{\aleph_1}, \aleph_2, 2^{\aleph_0})$ (Thm on Card. Addition)

$= 2^{\aleph_1}$ bec. $\aleph_2 \leq 2^{\aleph_1}$ & $2^{\aleph_0} \leq 2^{\aleph_1}$