MHF 4102 - AXIOMATIC SET THEORY FLORIDA INT'L UNIV.

TEST #1 - Spring '09 TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each question on a separate page.

(15) 1. Let $\langle A_i : i \in I \rangle$ be an indexed family of subsets of the set X. Prove that

$$\bigcap_{i \in I} (X - A_i) = X - \left(\bigcup_{i \in I} A_i \right).$$

(15) 2(a) Let $f: X \to Y$ be a function and suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. Define what are $f[A]$ and $f^{-1}[C]$.

(b) Is it always true that $f^{-1}[C - D] \subseteq f^{-1}[C] - f^{-1}[D]$?

(c) Is it always true that $f[A - B] \subseteq f[A] - f[B]$?

(20) 3(a) Write down the *Separation and Replacement axioms* in both their ordinary form and in class-form.

(b) Write down the *Union and Power Set axioms* in their ordinary form. Then translate them completely into the language of set theory.

(20) 4(a) Define *ordinal addition, multiplication,* and *exponentiation* by using transfinite recursion.

(b) Prove that $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ for all ordinals α, β, γ by using transfinite induction.

(15) 5(a) Prove that if $\alpha>0$ and $\beta<\gamma$, then $\alpha.\beta < \alpha.\gamma$ for all ordinals α, β, γ by transfinite induction on γ. [You may use the fact that "." is left distributive over "+" and that "+" & "." are associative, if needed.]

(b) Hence show that if $\alpha>0$ and $\alpha.\beta = \alpha.\gamma$, then $\beta = \gamma$.

(15) 6(a) Let $(A,<)$ be a partially ordered set. Define when $(A,<)$ is *well-founded* and define when $(A,<)$ is *well-ordered*.

(b) Write down what the Foundation Axiom says. Use it to prove that there are no sets y and z such that $y \in z$ and $z \in y$.

MHF - Axiomatic Set Theory — Florida Internat'l Univ.

Solutions to Test #1 — Spring 2009.

1. Let $a \in \bigcap_{i\in I}(X-A_i)$. Then $a \in X-A_i$ for each $i \in I$. So $a \in X$, and $a \notin A_i$ for each $i \in I$. $\therefore$ $a \in X$ and $a \notin \bigcup_{i\in I} A_i$.
$\therefore$ $a \in X-\left(\bigcup_{i\in I} A_i\right)$. Hence $\bigcap_{i\in I}(X-A_i) \subseteq X-\left(\bigcup_{i\in I} A_i\right) \ldots (*)$
Now let $a \in X-\left(\bigcup_{i\in I} A_i\right)$. Then $a \in X$ and $a \notin \left(\bigcup_{i\in I} A_i\right)$.
So $a \in X$ and $a \notin A_i$ for any $i \in I$. $\therefore$ $a \in (X-A_i)$ for each $i \in I$.
$\therefore$ $a \in \bigcap_{i\in I}(X-A_i)$. Hence $X-\left(\bigcup_{i\in I} A_i\right) \subseteq \bigcap_{i\in I}(X-A_i) \ldots (**)$
From $(*)$ & $(**)$, it follows that $\bigcap_{i\in I}(X-A_i) = X-\left(\bigcup_{i\in I} A_i\right)$.

2(a) $f[A] = \{f(a) : a \in A\}$, $\quad f^{-1}[C] = \{a \in X : f(a) \in C\}$

(b) YES. Let $a \in f^{-1}[C-D]$. Then $f(a) \in C-D$. So $f(a) \in C$ and $f(a) \notin D$. $\therefore$ $a \in f^{-1}[C]$ and $a \notin f^{-1}[D]$. Thus $a \in f^{-1}[C]-f^{-1}[D]$. Hence $f^{-1}[C-D] \subseteq f^{-1}[C]-f^{-1}[D]$.

(c) NO. Let $X = \{-2,0,2\}$, $Y = \{0,1,4\}$, $A = \{0,2\}$, $B = \{-2\}$ and $f : X \to Y$ be defined by $f(x) = x^2$. Then
$f[A-B] = f[\{0,2\}] = \{0,4\} \not\subseteq \{0\} = \{0,4\}-\{4\} = f[\{0,2\}]-f[\{-2\}] = f[A]-f[B]$.

3(a) Separation Axiom: If $\varphi(x)$ is any formula of LOST with one free variable x and A is a set then $\{x \in A : \varphi(x)\}$ is a set.
Class-form: If $\mathcal{C}$ is a class & A is a set, then $\mathcal{C} \cap A$ is a set.
Replacement Axiom: If $\varphi(x,y)$ is a function-type formula of LOST & A is a set, then $\{b : \varphi(a,b)$ holds for at least one $a \in A\}$ is a set.
Class-form: If $\mathcal{F}$ is a class-function & A is a set, then $\mathcal{F}[A]$ is a set.

(b) Union Axiom: If A is a set, then $\bigcup A$ is a set.
$(\forall x_1)(\exists x_2)(\forall x_3)\ \left((x_3 \in x_2) \leftrightarrow (\exists x_4)\left((x_3 \in x_4) \wedge (x_4 \in x_1)\right)\right)$
Power Set Axiom: If A is a set, then $\mathcal{P}(A)$ is a set.
$(\forall x_1)(\exists x_2)(\forall x_3)\ \left((x_3 \in x_2) \leftrightarrow (\forall x_4)\left((x_4 \in x_3) \rightarrow (x_4 \in x_1)\right)\right)$

4(a) Ordinal addition, multiplication, & exponentiation are defined by parametric recursion on β as follows:

(i) (a) $\alpha + 0 = \alpha$, (b) $\alpha + (\beta+1) = (\alpha+\beta)+1$, (c) $\alpha + \lambda = \sup\{\alpha+\beta : \beta<\lambda\}$

(ii) (a) $\alpha . 0 = 0$, (b) $\alpha.(\beta+1) = (\alpha.\beta)+\alpha$, (c) $\alpha.\lambda = \sup\{\alpha.\beta : \beta<\lambda\}$, and

(iii) (a) $\alpha^0 = 1$, (b) $\alpha^{\beta+1} = (\alpha^\beta).\alpha$, (c) $\alpha^\lambda = \sup\{\alpha^\beta : \beta<\lambda\}$.

Here α is the parameter and λ is a limit ordinal.

(b) We will prove the result by parametric transfinite induction on γ. So let α and β be arbitrary but fixed. For $\gamma = 0$, we have

$$\begin{aligned}(\alpha+\beta)+\gamma &= (\alpha+\beta)+0 = (\alpha+\beta) && \text{by (i)(a)}\\ &= (\alpha+(\beta+0)) = \alpha+(\beta+\gamma) && \text{by (i)(a) again.}\end{aligned}$$

So the result is true for $\gamma = 0$.

Assume that the result is true for γ. Then $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

$$\begin{aligned}\text{So } (\alpha+\beta)+(\gamma+1) &= ((\alpha+\beta)+\gamma)+1 && \text{by (i)(b)}\\ &= (\alpha+(\beta+\gamma))+1 && \text{by ind. hyp.}\\ &= \alpha+((\beta+\gamma)+1) && \text{by (i)(b) again}\\ &= \alpha+(\beta+(\gamma+1)) && \text{by (i)(b) once again.}\end{aligned}$$

So if the result is true for γ, it will be true for $\gamma+1$.

Finally assume that the result is true for all $\gamma<\lambda$, where λ is a limit ordinal. Then $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ for all $\gamma<\lambda$.

$$\begin{aligned}\text{So } (\alpha+\beta)+\lambda &= \sup\{(\alpha+\beta)+\gamma : \gamma<\lambda\} && \text{by (i)(c)}\\ &= \sup\{\alpha+(\beta+\gamma) : \gamma<\lambda\} && \text{by ind. hyp.}\\ &= \alpha+\sup\{\beta+\gamma : \gamma<\lambda\} && \text{by (i)(c) again}^*\\ &= \alpha+(\beta+\lambda) && \text{by (i)(c) once again.}\end{aligned}$$

So if the result is true $(\forall\gamma<\lambda)$, then it will be true for λ.

By the Principle of Transfinite Induction, the results follows for all γ. Since α and β were arbitrary, the results follow for all ordinals α, β, γ.

(Extra) (*) This works because $\delta = \sup\{\beta+\gamma : \gamma<\lambda\}$ is a limit ordinal.

$$\alpha+\delta = \sup\{\alpha+\zeta : \zeta<\delta\} = \sup\{\alpha+(\beta+\gamma) : \beta+\gamma<\delta\} = \sup\{\alpha+(\beta+\gamma) : \gamma<\lambda\}$$

5(a) We will prove the result by parametric Transf. Ind. on γ. So let α and β be arb. but fixed. Since $\beta < \gamma$, the least value of γ will be $\beta+1$. So we start the Ind. at $\gamma = \beta+1$.

For $\gamma = \beta+1$, we have

$$\alpha.\beta < \alpha.\beta + \alpha \quad \text{because } \alpha > 0$$
$$= \alpha.(\beta+1) = \alpha.\gamma \quad \text{by (b) of the def. of "."}$$

Hence the result is true for $\beta+1$.

Assume that the result is true for γ, where $\gamma \geq \beta+1$. Then $\beta < \gamma \Rightarrow \alpha.\beta < \alpha.\gamma$. Now suppose $\beta < \gamma+1$. Then either $\beta < \gamma$ or $\beta = \gamma$. If $\beta < \gamma$, then

$$\alpha.\beta < \alpha.\gamma < \alpha.\gamma + \alpha = \alpha.(\gamma+1) \text{ because } \alpha > 0.$$

And if $\beta = \gamma$, then $\alpha.\beta = \alpha.\gamma < \alpha.\gamma + \alpha = \alpha.(\gamma+1)$.

So $\beta < \gamma+1 \Rightarrow \alpha.\beta < \alpha.(\gamma+1)$. Hence if the result is true for γ, it will be true for $\gamma+1$.

Finally assume that the result is true for all $\gamma < \lambda$, where λ is a limit ordinal $\geq \beta+1$. Suppose now that $\beta < \lambda$. Then $\beta < \gamma_0$ for some $\gamma_0 < \lambda$. So

$$\alpha.\beta < \alpha.\gamma_0 \quad \text{because the result is true for } \gamma_0$$
$$\leq \sup\{\alpha.\gamma : \gamma < \lambda\} = \alpha.\lambda.$$

Hence if the result is true for all $\gamma < \lambda$, it will be true for λ.

By the Parametric Transfinite Induction Principle, the result follows for all α, β, and γ.

(b) Assume that $\alpha > 0$ and $\alpha.\beta = \alpha.\gamma$. Now suppose that $\beta \neq \gamma$. Then either $\beta < \gamma$ or $\gamma < \beta$. But if $\beta < \gamma$, then $\alpha.\beta < \alpha.\gamma$ (by part (a)) – contradicting $\alpha.\beta = \alpha.\gamma$. And if $\gamma < \beta$, then $\alpha.\gamma < \alpha.\beta$ (by part (a) again) – contradicting $\alpha.\beta = \alpha.\gamma$. So in either case we got a contradiction. Hence we must have $\beta = \gamma$.

So $(\alpha > 0) \wedge (\alpha.\beta = \alpha.\gamma) \Rightarrow \beta = \gamma$.

6(a) $\langle A, < \rangle$ is well-founded if every non-empty subset of A has a minimal* element.

$\langle A, < \rangle$ is well-ordered if every non-empty subset of A has a smallest** element.

(b) <u>Foundation Axiom</u>: If A is any non-empty set, then there is an $x \in A$ such that $x \cap A = \emptyset$.

Suppose there exist sets y and z such that $y \in z$ and $z \in y$. Let $A = \{y, z\}$. Then $z \in y \cap A$ because $z \in y$ and $z \in A$. Also $y \in z \cap A$ because $y \in z$ and $y \in A$. Since A has only two elements, namely y & z, it follows that there is no $x \in A$ such that $x \cap A = \emptyset$. But this contradicts the Foundation Axiom. Hence there are no sets y and z such that $y \in z$ and $z \in y$.

(Extra) (*) b is a minimal element of $B \subseteq A$ if there is no x in B with $x < b$. (x could be non-comparable with b)

(**) b is a smallest element of $B \subseteq A$ if for any $x \in B$, $x \leq b$.

A subset B can have many minimal elements but it can have only one smallest element.