MHF 4102 - AX. SET THEORY FLORIDA INT'L UNIV.

TEST #2 - Spring 2009 TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(20) 1. (a) Define what is *ordinal exponentiation* and what is the *cofinality* of a limit ordinal λ.

(b) Prove that for any ordinals α, β, γ we have $(\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}$. [You may use any results you need about ordinal arithmetic except this one, of course.]

(20) 2. Simplify the following *ordinal arithmetic* expressions as far as possible.

(a) $4 + \omega + \omega^2$ (b) $(\omega+2).(\omega+3)$ (c) $(\omega.2 + 1)^2$.

(15) 3. (a) Define what $A \prec B$ means and write down what the *Cantor-Bernstein Theorem* says.

(b) Let Q = Set of all rational numbers; and N = Set of all natural numbers. Prove that $Q \approx N$.

(15) 4 (a) Give the definitions of the following *cardinal operations*: $\kappa+\mu$, $\kappa.\mu$, κ^{μ}.

(b) Prove that for any two cardinals κ and μ, we always have (i) $\kappa+\mu = \mu+\kappa$ and (ii) $\kappa.\mu = \mu.\kappa$.

(15) 5. (a) Define what are *continuous* and *strictly increasing* class-functions from Ω to Ω. [Here Ω = class of all ordinals]

(b) Let A be any set and P(A) = the power set of A. Prove that $A \prec P(A)$.

(15) 6. (a) Write down what the Axiom of Choice says.

(b) Prove in ZF that $P(\omega)$, the power set of ω, can be *linearly ordered*. (*Hint*: *The set ω is well-ordered by $\in$.*)

MHF 4102 - Axiomatic Set Theory — Florida Int'l Univ.

Solutions to Test #2 — Spring 2009

1(a) (i) Ordinal exponentiation is defined by parametric transfinite recursion on β as follows: $\alpha^0 = 1$, $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, and $\alpha^\lambda = \sup\{\alpha^\beta : \beta < \lambda\}$ if λ is a limit ordinal.

(ii) The cofinality of a limit ordinal λ is the smallest ordinal θ such that we can find a sequence of ordinals $\langle \alpha_\beta : \beta < \theta \rangle$ with $\sup\{\alpha_\beta : \beta < \theta\} = \lambda$.

(b) We will prove $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$ by transfinite induction on γ. Let α and β be fixed and serve as parameters. If $\gamma = 0$, then we have $(\alpha^\beta)^\gamma = (\alpha^\beta)^0 = 1 = \alpha^0 = \alpha^{\beta\cdot 0} = \alpha^{\beta\cdot\gamma}$. So the result is true for $\gamma = 0$.

Now assume that the result is true for γ. Then $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

$$
\begin{aligned}
\text{So } (\alpha^\beta)^{\gamma+1} &= (\alpha^\beta)^\gamma \cdot (\alpha^\beta) && \text{by the def. of exp.} \\
&= \alpha^{\beta\cdot\gamma} \cdot \alpha^\beta && \text{by the ind. hyp.} \\
&= \alpha^{\beta\cdot\gamma+\beta} && \text{because } \alpha^\delta \cdot \alpha^\varepsilon = \alpha^{\delta+\varepsilon} \\
&= \alpha^{\beta(\gamma+1)} && \text{by the def. of mult.}
\end{aligned}
$$

So if the result is true for γ, it will be true for $\gamma+1$.

Finally assume that the result is true for all $\gamma < \lambda$ where λ is a limit ordinal. Then

$$
\begin{aligned}
(\alpha^\beta)^\lambda &= \sup\{(\alpha^\beta)^\gamma : \gamma < \lambda\} && \text{by the def. of exp.} \\
&= \sup\{\alpha^{\beta\cdot\gamma} : \gamma < \lambda\} && \text{by the ind. hyp.} \\
&= \sup\{\alpha^\delta : \delta < \beta\cdot\lambda\} = \alpha^{\beta\cdot\lambda} && \text{by def. of exp.}
\end{aligned}
$$

since $\beta\cdot\lambda$ is a limit ordinal. So if the result is true for all $\gamma < \lambda$, it will be true of λ.

By the Principle of Transfinite Induction, the result is true for all ordinals γ. Since α and β were arbitrary, it is also true for all α and β also.

2 (a) $4+\omega+\omega^2 = (4+\omega)+\omega^2 = \omega+\omega^2 = \omega.1+\omega.\omega$

$= \omega.(1+\omega) = \omega.\omega = \omega^2.$

(b) $(\omega+2).(\omega+3) = (\omega+2).\omega + (\omega+2).3$

$= \sup\{(\omega+2).n : n<\omega\} + (\omega+2)+(\omega+2)+(\omega+2)$

$= \sup\{\omega.n+2 : n<\omega\} + \omega+(2+\omega)+(2+\omega)+2$

$= \omega.\omega + \omega+\omega+\omega+2 = \omega^2+\omega.3+2$

(c) $(\omega.2+1)^2 = (\omega.2+1)(\omega.2+1) = (\omega.2+1).\omega.2 + \omega.2+1$

$= \sup\{(\omega.2+1).n : n<\omega\}.2 + \omega.2+1$

$= \sup\{(\omega.2n)+1 : n<\omega\}.2 + \omega.2+1$

$= (\omega.\omega).2 + \omega.2+1 = \omega^2.2+\omega.2+1$

3 (a) (i) $A \preccurlyeq B$ means there exists an injection from A to B

$A \approx B$ means there exists a bijection from A to B

$A \prec B$ means $A \preccurlyeq B$ and $A \not\approx B$.

(ii) The Cantor-Bernstein Theorem: $(A \preccurlyeq B \wedge B \preccurlyeq A) \Rightarrow A \approx B$.

(b) Clearly $\mathbb{N} \subseteq \mathbb{Q}$. So $\mathbb{N} \preccurlyeq \mathbb{Q}$ (just take $f: \mathbb{N} \to \mathbb{Q}$, $f(x)=x$ as your injection). Now any element of $\mathbb{Q}$ can be uniquely represented in the form $q = (-1)^k.m/n$ where $k = 0$ or 1, $m \in \mathbb{N}$, $n \in \mathbb{N}-\{0\}$, $g.c.d(m,n)=1$, and $k=0$ in case $m=0$. Define $f: \mathbb{Q} \to \mathbb{N}$ by $f((-1)^k.m/n) = 2^k.3^m.5^n$. Then f is clearly an injection. So $\mathbb{Q} \preccurlyeq \mathbb{N}$. Hence $\mathbb{Q} \approx \mathbb{N}$ by the Cantor-Bernstein Theorem.

4 (a) $\kappa+\mu = |\kappa\times\{0\} \cup \mu\times\{1\}|$, $\kappa.\mu = |\kappa\times\mu|$, $\kappa^\mu = |\mathcal{F}(\mu,\kappa)|$

(b) Define $f: \kappa\times\{0\} \cup \mu\times\{1\} \to \mu\times\{0\} \cup \kappa\times\mu$ by $f(\alpha,0) = \langle\alpha,1\rangle$ and $f(\beta,1) = \langle\beta,0\rangle$. Then f is a bijection. So

$\kappa+\mu = |\kappa\times\{0\} \cup \mu\times\{1\}| = |\mu\times\{0\} \cup \kappa\times\{1\}| = \mu+\kappa$

Define $g: \kappa\times\mu \to \mu\times\kappa$ by $g(\alpha,\beta) = \langle\beta,\alpha\rangle$. Then g is a bijection. So $\kappa.\mu = |\kappa\times\mu| = |\mu\times\kappa| = \mu.\kappa$.

5(a) A class function $f: \Omega \to \Omega$ is continuous if for any limit ordinal λ, $f(\lambda) = \sup\{f(\alpha) : \alpha < \lambda\}$. It is strictly increasing if $\alpha < \beta \Rightarrow f(\alpha) < f(\beta)$.

(b) Let $\varphi: A \to P(A)$ be defined by $\varphi(a) = \{a\}$. Then φ is clearly an injection. So $A \preccurlyeq P(A)$. Now suppose $A \approx P(A)$. Then we can find a bijection $f: A \to P(A)$. Let $D = \{x \in A : x \notin f(x)\}$. Then $D \subseteq A$. So $D \in P(A)$. Since f is a bijection we can find an $x_0 \in A$ such that $f(x_0) = D$. Now either $x_0 \in D$ or $x_0 \notin D$. But if $x_0 \in D$, then $x_0 \notin D$ (bec. $f(x_0) = D$). And if $x_0 \notin D$, then $x_0 \in D$ (bec. $f(x_0) = D$). So in either case we get a contradiction. Hence we cannot have $A \approx P(A)$. Thus $A \prec P(A)$.

6(a) Axiom of Choice: Let $\mathcal{A}$ be a set of pairwise disjoint non-empty sets. Then there exists a set M which contains exactly one element of each member of $\mathcal{A}$.

(b) Let A and B be distinct sets of $P(\omega)$. Then $A \neq B$. So $(A-B) \cup (B-A) \neq \emptyset$. Since ω is a well-ordered set and $(A-B)\cup(B-A)$ is a non-empty subset of ω, $(A-B)\cup(B-A)$ has a smallest element, x_0 say. Put

$A \prec B$ if $x_0 \in B$ and

$B \prec A$ if $x_0 \in A$.

Then "$\prec$" will be a linear ordering on $P(\omega)$. END.

[The linear order $\prec$ will look like:

$\emptyset \prec \cdots \{3\} \prec \cdots \{2\} \prec \cdots\cdots \{1\} \prec \cdots\cdots \{1,2\} \prec \cdots\cdots \{0\} \prec \cdots \{0,1\} \prec \cdots\cdots \prec \omega$

$\emptyset$ and ω will be the smallest and largest elements in $P(\omega)$ and between any two elements there will be infinitely many elements in between.]