1.16 Let $P : u = v_0, \ldots , v_k = v$ be a u - v geodesic in a connected graph G. Prove that $d(u,v_i) = i$ for each integer i with $1 \le i \le k$.

Proof. It suffices to show that if P, as above, is a u - v path, and if $d(u,v_j) = j$ fails for some integer j with $1 \le j \le k$, then P is not a u - v geodesic. Thus, suppose there is an integer j with $1 \le j \le k$ and $d(u,v_j) \ne j$. Plainly, v_0 is adjacent to v_1. Consequently, $d(v_0 ,v_1) = 1$. Thus, $1 < j \le k$. Since

$$u = v_0, \ldots , v_j$$

is a u - v_j path of length j, $d(u,v_j) < j$. If $j = k$, we are done, for it follows that the path P above is not a geodesic. Thus, suppose $j < k$. Then there is a u - v_j path with length $d(u,v_j)$ which, when followed by the v_j - v path of length $k - j$,

$$v_j, \ldots , v_k = v,$$

yields a u - v walk of length l with

$$l = d(u,v_j) + (k - j) < j + (k - j) = k.$$

From Theorem 1.6, the is a u - v path P' with length at most l. Thus P is not a u - v geodesic.//