10.4 Prove or disprove:

(a) If a planar graph contains a triangle, then its chromatic number is 3.

(b) If there is a 4-coloring of a graph G, then $\chi(G) = 4$.

(c) If it can be shown that there is not a 3-coloring of a graph G, then $\chi(G) = 4$.

(d) If G is a graph with $\chi(G) \le 4$, then G is planar.

Solution: All of these are false generally. For (a) K_4 is planar and has a sufficiency of triangles, but $\chi(K_4) = 4$. For (b), the most that you can correctly conclude is that $\chi(G) \le 4$. Plainly C_5 has a 4-coloring, but $\chi(C_5) = 3$. For (c) the most you can say is that $\chi(G) > 3$. There is no 3-coloring of K_n for $n > 3$ and $\chi(K_n) = n$. Finally, regarding (d), here are three nonplanar graphs with their chromatic numbers: $\chi(K_{3,3}) = 2$. If PG is the Peterson Graph, from Problem 10.2, $\chi(PG) = 3$. And finally, $\chi(K_{3,3,3,3}) = 4$. It's not hard to see that $K_{3,3,3,3}$ is not planar. Just count the edges and check that the size of the graph is larger than 30, that of a maximal planar graph with 12 vertices.//