2.12 Prove that if G is a graph of order n such that

$$\Delta(G) + \delta(G) \geq n - 1,$$

then G is connected and $\text{diam}(G) \leq 2$. Show that bound n - 1 is sharp.

Solution:

We shall first prove that G is connected. One of the things that will come out of this will be that $\text{diam}(G) \leq 4$. The next thing we shall do is show that the bound n - 1 is sharp, and next we shall provide a simple example of a connected graph satisfying the hypotheses with $\text{diam}(G) = 3$. This example is a result of solving Problem 2.10(a) and reveals that $\text{diam}(G) \leq 2$ is not a consequence of the hypotheses. Finally, we shall provide an example G of a connected graph of order 12 that satisfies the hypotheses and has $\text{diam}(G) = 4$. This, of course, shows that the upper bound on diameter of the elementary connectivity argument really cannot be improved.

To show G is connected, let u and v be vertices of G. If u and v are the same or adjacent, there is really nothing to do. So suppose that they are neither. There is a vertex, w, with $\deg(w) = \Delta(G)$. If either u or v is w, then we may proceed as in the proof of Theorem 2.4 to show there is a path with length at most 2 from u to v. Thus, suppose that neither u nor v is w. Since $\deg(u) + \deg(w) \geq \delta(G) + \Delta(G) \geq n - 1$, there must be a vertex, x, in G that is adjacent to both u and w. Also, since $\deg(v) + \deg(w) \geq \delta(G) + \Delta(G) \geq n - 1$, there is a vertex, y, in G that is adjacent to both v and w. Plainly,

$$W: u,x,w,y,v$$

is a walk of length 4 from u to v. From Theorem 1.6, there is a path from u to v with length at most 4. Consequently G must be connected with $\text{diam}(G) \leq 4$.

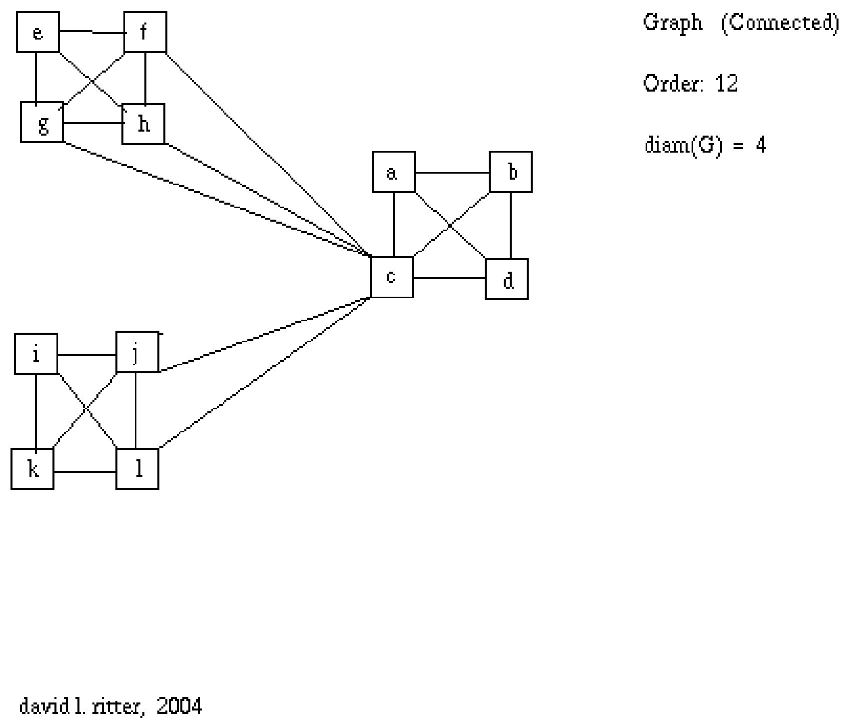
To see that the bound n - 1 is sharp, one need only consider the graph $G = 2K_k$ with $k \geq 3$. G has order $n = 2k$, two components, and $\Delta(G) = \delta(G) = k - 1$. So $\Delta(G) + \delta(G) = 2k - 2 = n - 2$.

Next, let's consider the little matter of "$\text{diam}(G) \leq 2$". Let H be the graph built from $G = 2K_k$ with $k \geq 3$ as follows: Choose a vertex from one of the K_k's of G, and label it u. Then choose a vertex from the other K_k and label it v. Add the edge uv. Then H is connected, $\deg(u) = \deg(v) = k$, and if x is a vertex different from u and v, $\deg(x) = k - 1$. Thus,

$$\Delta(H) + \delta(H) = k + (k - 1) = n - 1.$$

It is easy but slightlen tedious to verify that $\text{diam}(H) = 3$. [Draw a picture of the case k = 3!]

Finally, consider the connected graph G below constructed by beginning with 3 K_4's and then adding edges so as to obtain connectivity, but with enough care so that $\Delta(G) = \deg(c) = 8$, $\delta(G) = 3$, and $\operatorname{diam}(G) = 4 = d(e,i) = d(e,k)$.



Can you find an example of smaller order??