4.2 Prove that every connected graph all of whose vertices have even degrees contains no bridges.

Proof: It suffices to show that if a connected graph G has a bridge, then it must have an odd vertex. To this end, suppose that e = uv is a bridge in G. If either u or v is of odd degree, we have nothing to do. Thus, suppose that both u and v have even degrees. [Since the graph is connected, the degrees of u and v must be at least 2.]

Now e a bridge of G implies that G - e is the disjoint union of a component that contains u, and a component H that contains v. Clearly $\deg_H(v) = \deg_{G-e}(v) = \deg_G(v) - 1$, and if w is any vertex in H different from v, $\deg_H(w) = \deg_{G-e}(w) = \deg_G(w)$. Consequently, $\deg_H(v)$ is odd, and applying the 1st Theorem of Graph Theory to the subgraph H, H must have a vertex w different from v with odd degree in H. Note, however, $\deg_H(w) = \deg_{G-e}(w) = \deg_G(w)$. Thus, w is an odd vertex of G.//