4.26 Prove that an edge e of a connected graph G is a bridge if, and only if e belongs to every spanning tree of G.

Proof:

$\Rightarrow$: [Contrapositive] Suppose that e does not belong to every spanning tree of G. Let T be a spanning tree that does not contain e. Then the tree T is a spanning subgraph of G - e. It follows from Theorem 4.2 that if u and v are any two vertices of G - e, then there is a unique u - v path in T. This is also a u - v path in G - e. Thus, G - e is connected and e is not a bridge.

$\Leftarrow$: [Contrapositive] Suppose e is not a bridge. Then G - e is connected, and Theorem 4.10 implies that G - e has a spanning tree T. Since V(G) = V(G - e), T is a spanning tree of G, a spanning tree that does not contain e.