5.10 Prove that a connected graph G of size at least 2 is nonseparable if, and only if any two adjacent edges of G lie on a common cycle of G.

Proof: Evidently, if G has size at least 2, then G must have order at least 3.

$\Rightarrow$: Suppose that G is nonseparable and let e = uv and f = vw be arbitrary adjacent edges of G. Then, since the order of G must be at least 3, Theorem 5.7 implies there is a cycle C containing u and w. The cycle C contains two distinct u - w paths P and P′. At least one of them does not contain v. Say for definiteness it is P. Then P followed by the path w, v , u is a cycle in G containing e and f.

$\Leftarrow$: Suppose any two adjacent edges of G lie on a common cycle. Let v be an arbitrary vertex of G. Since G is connected, $\deg(v) \geq 1$. If $\deg(v) = 1$, then v is not a cut-vertex. Thus, we may assume $\deg(v) \geq 2$. Then (1) either all vertices of G adjacent to v lie in the same component of G - v, in which case G - v is connected and v is not a cut-vertex, or (2) there must be distinct neighboring vertices u and w adjacent to v with u and w lying in different components of G - v, in which case G - v is separated and v is a cut-vertex.

We shall show the second possibility cannot happen. Let u and w be any two neighbors of v. Then e = uv and f = vw are adjacent edges in G. By hypothesis there is a cycle C in G containing e and f. Observe that C cannot contain any other of the edges of G that may be incident with v. It follows that by removing v and the edges e and f from C, we obtain a u - w path P in G that is also a path in G - v. Thus, it is not possible to have u separated from w in G - v.

Thus, with the present hypotheses, the second possibility above cannot happen, and v cannot be a cut-vertex of G. Since v was arbitrary, G must be nonseparable.//