5.24 Let G be a graph of order n, and let k be an integer with $1 \le k \le n - 1$. Prove that if $\delta(G) \ge (n + k - 2)/2$, then G is k-connected.

Proof: We shall provide an indirect proof.

Suppose that G is not k-connected. Then $\kappa(G) < k$. Since $\kappa(G) \le n - 2$, G cannot be complete.

If G is not connected, then there must be a component of G with at most $n/2$ vertices if n is even and $(n - 1)/2$ vertices if n is odd. Suppose v is a vertex in such a component. If n is even, then $\deg(v) \le (n/2) - 1 = (n - 2)/2$. If n is odd, then we have $\deg(v) \le (n - 1)/2 - 1 = (n - 3)/2$. Thus, regardless of whether n is odd or even $\delta(G) \le (n/2) - 1 < (n + k - 2)/2$ if $1 \le k \le n - 1$.

Thus, suppose that G is connected. There must be a vertex-cut U of G with $|U| = \kappa(G)$. $G - U$ must be of order $n - |U|$ and not connected. It follows that there is a component of $G - U$ with at most $(n - |U|)/2$ elements. Let v be any vertex in this component. Plainly v could be adjacent to any of the members of U. On the other hand, v can be adjacent to at most $[(n - |U|)/2] - 1$ vertices in the component. [Consider the case-work in paragraph 2 above.] Thus,

$$\begin{aligned}
\deg(v) &\le |U| + [(n - |U|)/2] - 1 \\
&= (n + |U| - 2)/2 \\
&< (n + k - 2)/2.
\end{aligned}$$

Consequently, in this case, too, $\delta(G) < (n + k - 2)/2$. //

Note: We could first argue, of course, that G must be connected since if $1 \le k \le n - 1$, then $\delta(G) \ge (n + k - 2)/2 \ge (n - 1)/2$ and Corollary 2.5 can be applied. The end-game as found in paragraph 3 would be the some, however.