5.8 (a) Let G be a nontrivial connected graph. Prove that if v is an end-vertex of a spanning tree of G, then v is not a cut vertex of G.

Proof: [Contrapositive] Suppose that v is a cut vertex of G. Then Corollary 5.4 implies that there are vertices u and w in V(G) distinct from v and each other such that each u - w path in G contains v. Let T be any spanning tree in G. We shall show that v is not an end-vertex of T. To see this, observe that Theorem 4.2 implies that there is a unique u - w path in T. This is also a u - w path in G, and thus must contain v. Thus, it follows that we must have $\deg_T(v) \geq 2$. Thus v is not an end-vertex of T. Since T was an arbitrary spanning tree of G, v will not be an end-vertex of any spanning tree of G.//

(b) Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Proof: [Direct] By Theorem 4.10, G must have at least one spanning tree T. Theorem 4.3 implies that T must have at least two end-vertices. From Part (a), each of these must fail to be a cut-vertex of G.//

(c) Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

Proof: Observe that the tree T_0 consisting of v, together with all of the neighbors of v and the edges incident with v, is a subgraph of G. There is a maximal tree T in G containing T_0 as a subgraph.

We claim that T must be a spanning tree. Suppose not. Then there is at least one vertex w in G but not in T such that

$$d(w,T) = \min\{ d(w,u): u \in V(T) \}$$

is smallest amongst vertices w not T.

We claim that d(w,T) = 1. Suppose not. Then there is some u in T with d(u,w) = d(w,T) = k > 1. Let P: $u = v_0, v_1, \ldots, v_k = w$ be a u - w geodesic in G. If v_1 is in V(T), then u is not closest to w. On the other hand, if v_1 is not in V(T), then w doesn't give the smallest value of d(w.T) amongst vertices of G not in V(T). Thus, it must follow that d(w,T) = 1.

Now this allows us to contradict the presumed maximality of T, for the tree $T_1 = (V(T) \cup \{w\}, E(T) \cup \{ uw \})$, where $u \in V(T)$ satisfies d(u,w) = d(w,T) = 1, is a tree containing T_0 and properly containing T. Thus, T must in fact span G.//

(d) Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path. [Recall that if a tree contains a vertex of degree exceeding 2, then T has more than 2 end-vertices.]

Proof: Suppose G is a connected graph with exactly two vertices, u and w, that are not cut-vertices.

First, if $|V(G)| = 2$, G must be isomorphic to $K_2 = P_2$. Thus, we may assume that G is of order at least 3 in the following.

Next, we claim $\Delta(G) = 2$. Since $|V(G)| \geq 3$ and G is connected, $\Delta(G) \geq 2$ may be easily seen to be true by considering any spanning tree of G. To see that $\Delta(G) \geq 3$ is untenable, observe that if $\Delta(G) \geq 3$, then from Part (c), there is a spanning tree T of G with $\Delta(T) \geq \Delta(G)$. Exercise 4.19 implies that T has at least 3 end-vertices, which cannot be cut-vertices of G from Part (a), above.

Since G is connected, $\deg(u) \geq 1$ and $\deg(w) \geq 1$. If either the degree of u or of w is greater than 1, then from Part (c), G would have a spanning tree without one of u or w being an end-vertex. This, however, would imply that G had a vertex different from u or w that is not a cut-vertex, which is impossible. Thus, we must have $\deg(u) = \deg(w) = 1$. Moreover, that u and w are the only vertices of G that are not cut-vertices implies that no other vertices of G may be of degree 1.

At this point, we know that G is connected, has order at least 3, has two vertices u and w of degree 1 that are the only vertices that are not cut-vertices of G, and that the remainder of the vertices are of degree 2. To see that G is a path, it suffices to show that a u - w geodesic

$$P: u = v_1, \ldots , v_k = w$$

from u to w, where $k \geq 2$ contains all the vertices of G. If P does not contain all the vertices of G, though, there are two problems we encounter. Either G is not connected, or there is some vertex not on P that is adjacent to at least one of the vertices of P, an impossibility in light of the known values for the degrees of the vertices of P. Thus P must contain all the vertices of G. Finally, G can have no edges except those appearing in P due to the degree values of the vertices of G. and thus G is isomorphic to P_k where $k = |V(G)|$. //