6.20 Let G be a graph of order $n \geq 3$ having the property that for each $v \in V(G)$, there is a Hamiltonian path with initial vertex v. Show that G is 2-connected (connected, order at least 3, and no cut-vertices.) but not necessarily Hamiltonian.

Proof: Suppose that the order of G is at least 3. We shall show that if G is not 2-connected, then there is a vertex $v \in V(G)$ that cannot be the initial vertex of any Hamiltonian path in G. Evidently, if G is not connected, there is really nothing to show since then G has no Hamiltonian paths and any vertex will do. Thus, suppose that G is connected but not 2-connected. Since the order of G is at least 3, G must have a cut-vertex v. From Corollary 5.4 it follows that there are two vertices u and w of G distinct from v such that v lies on every u - w path. In particular, for any u - w path, v must be an internal vertex. This implies, of course, that if G has a Hamiltonian path, then for any such path, v must be an internal vertex, not an initial vertex.

There remains the matter of G not necessarily being a Hamiltonian graph. For this an example suffices. Let G be the Peterson graph, PG, of Figure 6.11. PG is not Hamiltonian, but it is easy to find Hamiltonian paths initiating at any vertex. For example, if you begin on a vertex that is on the pentagon, say $u_1$, you merely cycle clockwise around the pentagon until you reach $u_5$. Then follow this by $v_5$, and the continue around the pentacle $v_3$, $v_1$, $v_4$, $v_2$. You can do something similar if you begin on a pentacle vertex. Traverse the vertices on the pentacle. [Remember the pentacle is really $C_5$ in disguise.] Then go traipsing around the pentagon --- cyclically, of course.//