7.2 Prove that a graph G has an Eulerian orientation if, and only if G is Eulerian.

Proof:

$\Rightarrow$: [Contrapositive] It suffices to ignore trivial and disconnected grgaphs. Thus, suppose that G is a connected graph that is not Eulerian. Then G has a vertex $v \varepsilon V(G)$ with $\deg_G(v)$ an odd integer. Let D be any digraph with $V(D) = V(G)$ obtained by providing an orientation to the edges of G. In short, let D be any orientation of G. Then there are a total of $\deg_G(v)$ arcs in $E(D)$ with v as either initial element of the pair or terminal element of the pair. Since there must be an odd number of these arcs, it follows that $id_D(v) \neq od_D(v)$. From Theorem 7.4, the digraph D cannot be Eulerian. Since D was an arbitrary orientation of G, G cannot have an Eulerian orientation.

$\Leftarrow$: [Direct Proof] Suppose that G is a Eulerian graph. Then G has an Eulerian circuit, say

$$C: \quad u = v_0, \ \ldots \ , v_k = u.$$

Let D be the digraph with vertex set $V(D) = V(G)$ and whose arc set is $E(D) = \{ (v_{i-1}, v_i) : i = 1, \ldots, k \}$ with the v_i 's from the Eulerian circuit C above. Then D is an orientation for G since each edge of G appears exactly once in C, and D is Eulerian due to C actually being an Eulerian circuit for D as well for G.//