7.6 Does there exist a nontrivial digraph D in which no two vertices of D have the same outdegree, but every two vertices have the same indegree?? Explain.

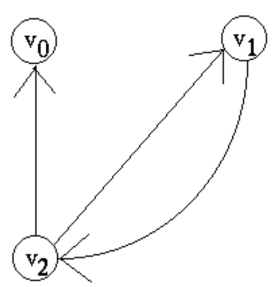
Solution: The first question we need to answer is this:

If there is such a digraph D, what follows necessarily from the degree restrictions??

Suppose that D is such a digraph of order n with vertex set given by $V(D) = \{v_1, ..., v_n\}$. From the 1st Theorem of Digraph Theory, it follows that

$$\begin{aligned} n\, id(v_1) &= \sum_{v \in V(D)} id(v) \\ &= \sum_{v \in V(D)} od(v) \\ &= \sum_{i=0}^{n-1} i \\ &= \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}. \end{aligned}$$

We can read a couple of things from this equation. The first is that if there is such a digraph D, then the common indegree must be $id(v_1) = (n-1)/2$, and the second is that such a digraph must be of odd order.



D_1

It is not terribly difficult to create an example of a digraph D_1 of order 3 with vertex set $V(D) = \{v_0,v_1,v_2\}$, $id(v) = 1$ for every vertex, and $od(v_0) = 0$, $od(v_1) = 1$, and $od(v_2) = 2$. [Consider the digraph, D_1, to the left.] A little thought reveals that there is, in fact, an infinite sequence of such examples. We shall show how to build them recursively, via the proof of the following theorem:

Theorem. For each positive integer $k \geq 1$, there is a digraph D_k of order 2k+1 with $V(D_k) = \{v_0,v_1, ..., v_{2k}\}$ such that $id(v) = k$ for every vertex v, and $od(v_j) = j$ for $j = 0,...,2k$.

Proof: First, we have the basis for a recursive construction in D_1 above. Thus, for our induction hypothesis suppose that k is an arbitrary positive integer and that we already have a digraph D_k of order 2k+1 with vertex set $V(D_k) = \{v_0,v_1, ..., v_{2k}\}$ such that $id(v) = k$ for every vertex v, and $od(v_j) = j$ for $j = 0,...,2k$.

We shall now show how to build D_{k+1} by using D_k. Create two new vertices y and z, and let D_{k+1} have vertex set

$$V(D_{k+1}) = V(D_k) \cup \{y,z\}.$$

We now have the necessary 2(k+1) + 1 vertices.

Connecting the vertices is slightly stickier. We shall begin with the arcs of D_k fixed. Then, by connecting the vertex z to each vertex in $V(D_k)$ and to y, we up the indegree of these vertices from k to k+1 and have the indegree of y at 1. Ignoring the vertex v_0, we use a partition of the 2k other vertices of D_k into two sets of k elements to raise the indegrees of y and z by k. In the case of y, we reach k+1, but the indegree of z is only at k. This also raises the outdegrees of v_1 through v_{2k} by 1. So we no longer have a vertex of outdegree 1. We thus send an arc from y to z. This raises the indegree for z to k+1 and gives us a vertex with outdegree 1. In short, we set

$$E(D_{k+1}) = E(D_k) \cup E_1 \cup E_2 \cup E_3$$

where

$$E_1 = \{ (z,v_j) : 0 \le j \le 2k \} \cup \{ (z,y) \},$$

$$E_2 = \{ (v_j,y), (v_{k+j},z): 1 \le j \le k \}, \text{ and}$$

$$E_3 = \{ (y,z) \}.$$

It then follows from the induction hypothesis and the construction above that we have

$$id(v) = k+1 \text{ for every vertex } v \in V(D_{k+1}),$$

$$od(v_0) = 0,$$

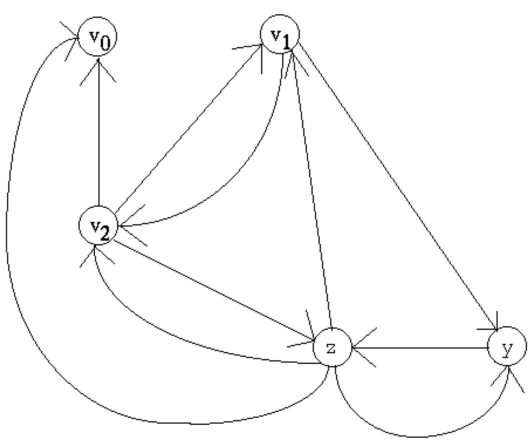
$$od(y) = 1,$$

$$od(v_j) = j+1 \text{ for each } j \text{ with } 1 \le j \le 2k,$$

and

$$od(z) = 2(k+1).$$

All that is left to do is to re-label the vertices.[The result of going from k = 1 to k = 2 without re-labelling is below.]//



Question: What happens when you replace "nontrivial digraph" in the original question with "orientation of a nontrivial graph"?? Proof???