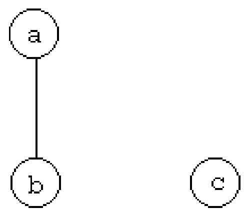
8.16 Prove that if G is a graph of order n and maximum degree Δ, then $\alpha(G) \geq n/(\Delta + 1)$.

Remark. Obviously $\Delta = \Delta(G)$ and the graph G must not have any isolated vertices. Although a minimum vertex cover may be defined for any nontrivial graph G, isolated vertices are a major annoyance. Of course, $\alpha_1(G)$ is not defined if G has isolated vertices. Here's an example:



The graph to the left plainly has $\alpha(G) = 1$, $\Delta(G) = 1$, and order $n = 3$. Consequently, $\alpha(G) < 3/2 = n/(\Delta + 1)$. Ouch.

Proof: Suppose that U is a set of vertices that cover all the edges of G with $\alpha(G) = |U|$. Let $W = V(G) - U$. Clearly, we have

(1) $$n = |U| + |W|.$$

Since G does not have any isolated vertices, each vertex $w \varepsilon W$ must be adjacent to at least one vertex $u \varepsilon U$. Thus

(2) $$\begin{aligned} |W| &\leq \left| \bigcup_{u \in U} N(u) \right| \\ &\leq \sum_{u \in U} |N(u)| \\ &\leq \sum_{u \in U} deg_G(u) \\ &\leq |U|\Delta \\ &\leq \alpha(G)\Delta. \end{aligned}$$

Evidently, (1) and (2) imply that

$$n \leq \alpha(G)(\Delta + 1)$$

which is equivalent to the conclusion.//

Question: It's not hard to show that $\alpha(G) \geq \beta_1(G)$. Can you improve this by proving $\beta_1(G) \geq n/(\Delta + 1)$??