8.4 A connected bipartite graph G has partite sets U and W, where $|U| = |W| = k \geq 2$. Prove that if every two vertices of U have distinct degrees in G, then G contains a perfect matching.

Proof: We shall use Theorem 8.3, naturally. We shall show that U is neighborly. To this end, let X be an arbitrary nonempty subset of U. Suppose $|X| = j \geq 1$. Since G is connected, for each $v \in X$, $\deg_G(v) \geq 1$. Since

$$N(X) = \bigcup_{v \in X} N(v),$$

it follows that for each $v \in X$, $\deg_G(v) = |N(v)| \leq |N(X)|$. To finish, we need only see that there is some vertex $v \in X$ with $j \leq \deg_G(v)$. Suppose, to the contrary, that we actually have $j > \deg_G(v)$ for each vertex $v \in X$. Then, of course, we get in trouble. The set of possible values for the degrees of the j vertices in X consists of the integers 1, ..., j-1. The Pigeon Hole Principle then implies there are two vertices u and v in X with $\deg_G(u) = \deg_G(v)$. Oops. Thus, there must be some vertex $v \in X$ such that $|X| = j \leq \deg_G(v) = |N(v)| \leq |N(X)|$. It follows that U is neighborly and thus Hall's Theorem 8.3 implies that G has a matching of cardinality k, as required. //

[The order of G is 2k. So a perfect matching must be of cardinality k.]