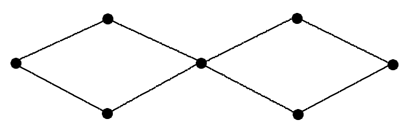
Exercise 2.19. Prove that if G is a graph of order n such that $\Delta(G)+\delta(G) \geq n-1$, then G is connected and $\text{diam}(G) \leq 4$. Show that the bound $n-1$ is sharp.

Proof. Let u and v be two nonadjacent vertices of G. It suffices to show that there is a $u-v$ walk of length at most 4 in G. If $\deg u + \deg v \geq n-1$, then there is a vertex in G that is adjacent to both u and v and so G contains a $u-v$ path of length 2. Thus we may assume that $\deg u + \deg v < n-1$. Hence $\deg u < \Delta(G)$ and $\deg v < \Delta(G)$. Let $w \in V(G)$ with $\deg w = \Delta(G)$. Then

$$\deg u + \deg w = \deg u + \Delta(G) \geq \delta(G) + \Delta(G) \geq n-1$$

and so there exists $x \in V(G)$ such that x is adjacent to u and w. Similarly, $\deg w + \deg v \geq n-1$ and so there exists $y \in V(G)$ such that y is adjacent to w and v. Therefore, u, x, w, y, v is a $u-v$ walk of length 4 in G. For $n = 2k$, $k \geq 1$, the graph $2K_k$ shows that the bound $n-1$ is sharp. ■

Observe that the graph G below has order 7 (minimum), $\Delta(G) = 4$, $\delta(G) = 2$, and $\text{diam}(G) = 4$. Also, the graph H below has order 9, $\Delta(G) = 6$, $\delta(G) = 2$, and $\text{diam}(G) = 4$.



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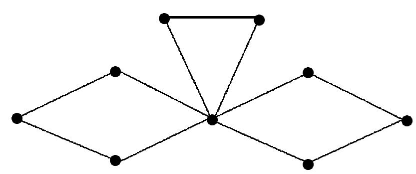

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