20. Let C be a collection of sets and E an element in the σ-algebra generated by C. Then there is a countable subcollection $C_0 \subset C$ such that E is an element of the σ-algebra A_0 generated by C_0.

Proof: Let's denote the σ-algebra generated by a collection B of subsets of X by $\sigma(B)$. This means, for example, that $E \in \sigma(C)$. Now let $A' = \cup\{\sigma(B): B \subset C$ and B is countable$\}$. A little thought reveals that $A' \supset C$. To finish things, it suffices to show that A' is a σ-algebra, for then it follows that $E \in \sigma(C) \subset A'$, and thus, that there is a countable subset C_0 of C with $E \in A_0 = \sigma(C_0)$ from the definition of a union of a family of sets.

To show A' is a σ-algebra, it suffices to show that A' is closed with respect to complementation and countable unions. That A' is closed under complements is simple because if $F \in A'$, there is a countable subset B of C such that $F \in \sigma(B)$. Since $\sigma(B)$ is an algebra, $\sim F \in \sigma(B) \subset A'$. That A' is closed under countable intersections requires a little more work. Let $\{E_i: i \in I\}$ be a countable collection of subsets of A'. For each i, there is a countable subset B_i of C such that $E_i \in \sigma(B_i)$, the σ-algebra generated by B_i. Let $B' = \cup\{B_i: i \in I\}$. Since B' is a countable union of countable subsets of C, B' is a countable subset of C. Hence, $\sigma(B') \subset A'$. Because $B_i \subset B'$ for each i, $E_i \in \sigma(B_i) \subset \sigma(B')$ for each i. Consequently, $\cup\{E_i: i \in I\} \in \sigma(B') \subset A'$, and we are finished.■