32. Let Y be the set of ordinals less than the first uncountable ordinal; i.e., $Y = \{x \varepsilon X : x < \Omega \}$. Show that every countable subset E of Y has an upper bound in Y and hence a least upper bound.

Proof: Let's denote the order on Y inherited via Proposition 8 of Chapter 1 by <. Suppose that E is a countable, non-empty subset of Y. For each $y \varepsilon E$, let

$$S_y = \{ x \varepsilon Y: x \leq y \} = \{ x \varepsilon Y: x < y \} \cup \{ y \}.$$

From Proposition 8 of Chapter 1, S_y must be countable for each $y \varepsilon E$. Thus, $S = \cup \{S_y : y \varepsilon E \}$, a countable union of countable sets, must be countable. From the proof of Proposition 8 of Chapter 1, Y must be uncountable. Consequently, $Y \sim S$ must be non-empty. Observe that from the simplicity of the well ordering and our friendly De Morgan's laws, $Y \sim S = \cap \{ T_y : y \varepsilon E \}$, where $T_y = \{ x \varepsilon Y : x > y \}$ for each $y \varepsilon E$. Thus, if $y_0 \varepsilon Y \sim S$ and $y \varepsilon E$, then $y < y_0$. Thus, any element of $Y \sim S$ will be an upper bound for E. Evidently, from the well ordering, the set of upper bounds must have a least element, and we are finished.■