33. A subset S of a well-ordered set X is called a segment if

$$S = \{ x \in X : x < y \}$$

for some $y \in X$ or $S = X$. Show that a union of segments is again a segment.

Proof. Suppose σ is a collection of segments of X. If σ is empty, then $\cup\sigma = \emptyset = \{ x \in X : x < y_0 \}$, where y_0 is the least element of X under the well-ordering. Likewise, if $\cup\sigma = X$, we have nothing to do. Thus, we may consider the case where $X \sim \cup\sigma \neq \emptyset$ and $\cup\sigma$ is non-empty. In this case, none of the segments in σ may be X, and consequently, there must be a non-empty subset E of X such that $\sigma = \{ \{ x \in X : x < y \} : y \in E \}$. Let y_0 be the least element of the non-empty set $X \sim \cup\sigma$. We claim now that

$$\cup\sigma = \{ x \in X : x < y_0 \}.$$

First, $X \sim \cup\sigma = \cap\{ X \sim \{ x \in X : x < y \} : y \in E \}$ from our friendly De Morgan's laws. A routine argument using this reveals that we must have $\{ x \in X : x \geq y_0 \} \subset X \sim \cup\sigma$. Thus,

$$\cup\sigma \subset \{ x \in X : x < y_0 \}.$$

The containment cannot be proper, for otherwise, we have a contradition to y_0 being the least element of $X \sim \cup\sigma$. Thus, we must have equality, and we are finished.■