21. Let E be a set of positive real numbers. We define $\sum_{x \varepsilon E} x$ to be $\sup\{ s_F : F \varepsilon \mathscr{F} \}$, where $\mathscr{F}$ is the collection of finite subsets of E, and s_F is the (finite) sum of the elements of F.

a. Show that $\sum_{x \varepsilon E} x < \infty$ only if E is countable.

b. Show that if E is countable and $\langle x_n \rangle$ is a one-to-one mapping of $\mathbb{N}$ onto E, then $\sum_{x \varepsilon E} x = \sum_{n=1}^{\infty} x_n$.

Proof. (a) Suppose that E is a set of real numbers with $\sup\{ s_F : F \varepsilon \mathscr{F} \} < \infty$, where $\mathscr{F}$ is the collection of finite subsets of E, and s_F is the (finite) sum of the elements of F. Let S denote this supremum. For each $n \varepsilon \mathbb{N}$, let

$$E_n = \{ x \varepsilon E: x \geq 1/n\}.$$

We shall show that E_n is finite for each $n \varepsilon \mathbb{N}$. Since we plainly have $E = \cup\{E_n: n \varepsilon \mathbb{N}\}$, this will imply that E is countable.

Let $n \varepsilon \mathbb{N}$ be fixed. Suppose now that F is an arbitrary finite subset of E_n. Then we must have $|F| \cdot (1/n) \leq s_F \leq S$, where $|F|$ denotes the number of elements in F. It follows that we have $|F| \leq n \cdot S$ for any finite subset F of E_n. Thus E_n must be finite and cannot contain more than $n \cdot S$ elements. Thus, we are finished proving Part (a).//

(b) From Problem 2-18, and its proof, it is evident that

$$T = \sum_{n=1}^{\infty} x_n$$

exists either as a positive real number or ∞, and is, in fact, the supremum of the increasing sequence of partial sums. Since each partial sum is a finite sum of the sort used to define

$$S = \sum_{x \varepsilon E} x ,$$

it follows that $T \leq S$. To see that $T < S$ is not possible, it suffices to observe the following: If F is a finite subset of E, then there is a largest index J of the elements $x_n \varepsilon F$, and if s_J is the partial sum with index J, then $s_F \leq s_J$, since the partial sum is over all of the members of the sequence with index no larger than J.//