43. Let f be that function defined by setting

$$f(x) = \begin{cases} x & , \quad \text{if } x \text{ is irrational} \\ p \cdot \sin(1/q) & , \quad \text{if } x = p/q \text{ in lowest terms.} \end{cases}$$

At what points is f continuous?

Solution. We shall show that f is continuous at each irrational and at $x = 0$, and that f fails to be continuous at each rational number different from 0.

First, we need to recall a few pieces of trivia regarding our friendly sinus problem. Recall that if we have $0 < x < \pi/2$, then $\sin(x) < x$. Thus,

(1) $$\sin(1/q) < 1/q \text{ for } q \in \mathbb{N} = \{1,2,3,...\}.$$

We shall also need to dig a bit deeper to Taylor a nice estimate to our purposes. From elementary Calculus, if $0 < x$, we have $\sin(x) = x - \sin(\xi)(x^2/2)$ for some real number ξ with $0 < \xi < x$. Consequently, we easily obtain

(2) $$0 < (1/q) - \sin(1/q) \leq (1/2)\cdot(1/q)^2 \text{ for } q \in \mathbb{N}.$$

The second thing worth noting is that f is odd. Here, all that we need check is when $x = p/q$ in lowest terms. Since $-x = (-p)/q$, $f(-x) = -p\cdot\sin(1/q) = -[p\cdot\sin(1/q)] = -f(x)$. As a consequence, with the exception of the matter of continuity at zero, we may restrict our attention to the positive real numbers when treating questions of continuity at particular points.

Now let's deal with continuity at $x = 0$. From the definition of f, $f(0) = 0$. From inequality (1), it follows that we have $|f(x) - f(0)| \leq |x - 0|$, from which the continuity at zero follows trivially.

We now restrict our attention to the positive real numbers.

Suppose that $y > 0$ is rational. We may suppose that $y = p/q$ is in lowest terms with both p and q positive integers. From inequality (1), it follows that $\varepsilon_0 = (y - f(y))/2$ is positive. We shall show that for each $\delta > 0$ there is a real number x with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$, and thus, that f is not continuous at y. To this end, suppose that $\delta > 0$. There is an irrational number x with $y < x < y + \min(\delta, \varepsilon_0)$. Evidently, $|x - y| = x - y < \delta$ and $|x - y| < \varepsilon_0$. Thus,

$$\begin{aligned} |f(x) - f(y)| &\geq |y - f(y)| - |y - f(x)| \\ &\geq |y - f(y)| - |y - x| \\ &\geq 2\varepsilon_0 - |y - x| \geq \varepsilon_0. \end{aligned}$$

Since $\delta > 0$ was arbitrary, we have established that f is not continuous at y when $y > 0$ is rational.

Finally, suppose that $y > 0$ is irrational. We must show that for each $\varepsilon > 0$, there is a real number $\delta > 0$ so that if x is any real number with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$, in order to verify the continuity of f at y.

To this end, suppose that $\varepsilon > 0$. There is a unique positive integer n_0 such that $n_0 - 1 < y < n_0$. From the Axiom of Archimedes, there is a positive integer N_0 such that if $q \in \mathbb{N}$ and $q \geq N_0$, then $n_0/q < \varepsilon$. Observe that the set of rational numbers, p/q, in lowest terms in the interval $(n_0 - 1, n_0)$ with p and q positive and q smaller than N_0 is finite. Choose δ now to be the minimum of the distance from y to this set of points and $\varepsilon/2$. Since y is irrational and the set of points is finite and consists of rational numbers, $\delta > 0$.

We'll now verify that this magical δ does the job. Suppose x is a real number satisfying $|x - y| < \delta$. If x is irrational, then $|f(x) - f(y)| = |x - y| < \varepsilon/2 < \varepsilon$. If x is rational, then from the definition of δ, we may assume that $x = p/q$ with p and q positive integers with no common factors, except the usual trivial ones, and $q \geq N_0$. Then, using inequality (2), we have

$$\begin{aligned} |f(x) - f(y)| &\leq |p \cdot \sin(1/q) - (p/q)| + |(p/q) - y| \\ &< [(p/q) - p \cdot \sin(1/q)] + \varepsilon/2 \\ &< p \cdot (1/2) \cdot (1/q)^2 + \varepsilon/2 \\ &< q n_0 \cdot (1/2) \cdot (1/q)^2 + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$

after flogging a deceased varmint, and we are done.